FALL 2014

1. Let A be a subset of $[0,1]$. Let m^* be Lebesgue outer measure on $[0,1]$.
(a) State the definition of "Lebesgue measurable set".
(b) Show that A is Lebesgue measurable if and only if

$$m^*(A) + m^*(A^c) = 1,$$

where A^c is the complement of A in $[0,1]$.

a.) a set E is Lebesgue measurable if
$m^*(A) = m^*(E \cap A) + m^*(E^c \cap A) \quad \forall E \in Bor(\mathbb{R})$
where m^* is Lebesgue outer measure

Lebesgue outer measure is defined
$m^*(E) = \inf\{\sum_i |b_i - a_i| \mid E \subset \bigcup_i (a_i, b_i)\}$

b.) ($\Rightarrow$) assume A is Leb measble, i.e. $m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$
$\forall E \in Bor(\mathbb{R})$. Then let $E = [0,1]$, get
$m^*([0,1]) = m^*(\underbrace{[0,1] \cap A}) + m^*(\underbrace{[0,1] \cap A^c})$

$A \subseteq [0,1]$ so $[0,1] \cap A = A$ $\qquad$ $A^c \subseteq [0,1]$ so $[0,1] \cap A^c = A^c$

$\Rightarrow 1 = m^*(A) + m^*(A^c)$ ✓

($\Leftarrow$) assume $1 = m^*(A) + m^*(A^c)$

CR: AJ

Folland pg. 32

19. Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the **inner measure** of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$. (Use Exercise 18.)

on an algebra
$\mu_0(\emptyset) = 0$ and
ctbly add on
disjt sets

WTS: $m^*(A) = m_*(A)$
$m_*(A) := m_0(X) - m^*(A^c)$
$\quad //$
$X = [0,1]$ so m_0 is finite
and $m_0([0,1]) = 1$

Then $m_*(A) = 1 - m^*(A^c)$
$= m^*(A)$ ✓ (by assumption)

2. Let f be a Lebesgue measurable function on $\mathbb{R}$ such that both

$$\int_{\mathbb{R}} |f(x)|\,dx < \infty \quad \text{and} \quad \int_{\mathbb{R}} |xf(x)|\,dx < \infty.$$

Define the function $F(\xi)$ by

$$F(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x)\,dx, \quad \xi \in \mathbb{R}.$$

Show that F is differentiable at each $\xi \in \mathbb{R}$. WTS: deriv exists (is finite) $\forall \xi \in \mathbb{R}$

CR: RILEY

mimic proof of Folland Thm. 2.27b

Check: $\frac{d}{d\xi} F(\xi) = \lim_{h\to 0} \frac{F(\xi+h) - F(\xi)}{h \leadsto = (\xi+h) - \xi}$

$F(\xi+h) - F(\xi) = \int e^{i\xi(x+h)} f(x+h) - e^{i\xi x} f(x)\,dx$

$\lim_{h\to 0} \frac{1}{h}(F(\xi+h) - F(\xi)) = \lim_{h\to 0} \int \frac{1}{h}\left(e^{i(\xi+h)x} f(x+h) - e^{i\xi x} f(x)\right) dx$

↳ want to be able to pull limit inside to get $\int \lim_{h\to 0} \frac{1}{h}(e^{i(\xi+h)x} f(x) - e^{i\xi x} f(x))\,dx$

$\int \lim_{h\to 0}(ix\,e^{i(\xi+h)x} f(x))\,dx$ (L'Hopital)

$\int ix \cdot e^{i\xi x} f(x)\,dx$

↳ note this exists as $|ix \cdot e^{i\xi x} \cdot f(x)|$

$= |ix| \cdot |e^{i\xi x}| \cdot |f(x)|$

$= |x| \cdot 1 \cdot |f(x)| = |x f(x)|$

and $\int |x f(x)|\,dx < \infty$ (given)

So let's back up & show we can use DCT to pull in limit!

- Define $\{f_n\}$ so that $\lim \int f_n = \int \lim f_n = \int f$

  $\{f_n\} = 1/h\,(e^{i(\xi+h)x} f(x) - e^{i\xi x} f(x))$

  let $h = 1/n$, then $\lim_{h\to 0} = \lim_{n\to\infty}$ ✓

  $\{f_n\} \equiv n(e^{i(\xi+1/n)x} f(x) - e^{i\xi x} f(x))$

- check $|f_n| \le g$ for some $g \in L^1$

  $h(\xi) = e^{i\xi x} f(x)$ $h'(\xi) = ix e^{i\xi x} f(x)$ so h is cont + diff'able

  Then apply MVT: $h'(\xi_0) = \frac{h(\xi+1/n) - h(\xi)}{(\xi+1/n) - \xi}$ for some ξ_0 $\Rightarrow ixe^{i\xi_0} f(x) = n(e^{i(\xi+1/n)x} - e^{i\xi x}) f(x)$

  $\Rightarrow \underbrace{\sup_{\xi_0 \in \mathbb{R}} |h'(\xi_0)|}_{= g} \ge |n(e^{i(\xi+1/n)x} - e^{i\xi x}) f(x)|$ ✓

- Check $\{f_n\} \to f$ ptwise. $|(\xi + \frac{1}{n}) - \xi| < \delta \Rightarrow |1/n| < \delta \Rightarrow |1/n - 0| < \delta$

  * could probably claim this. Or actually finish proving it

3. If E_1 and E_2 are two nonempty sets in $\mathbb{R}^2$, define

$$d(E_1, E_2) = \inf_{x_1 \in E_1, x_2 \in E_2} \rho(x_1, x_2)$$

where ρ is the standard Euclidean metric.

(a) Give an example of disjoint nonempty closed sets in $\mathbb{R}^2$ with $d(E_1, E_2) = 0$.

(b) Let E_1, E_2 be nonempty sets in $\mathbb{R}^2$ with E_1 closed and E_2 compact. Show that there exist $x_1 \in E_1$ and $x_2 \in E_2$ such that $d(E_1, E_2) = \rho(x_1, x_2)$. Deduce that $d(E_1, E_2) > 0$ if such E_1, E_2 are disjoint.

CR: RILEY

a.) $E_1 = \{(n, 0) \mid n \in \mathbb{N}\}$
$E_2 = \{(n + \frac{1}{2n}, 0) \mid n \in \mathbb{N}\}$

$$\begin{aligned} d(E_1, E_2) &= \inf \rho(x_1, x_2) \\ &\leq \inf_{n \in \mathbb{N}} |n - (n + \tfrac{1}{2n})| \\ &= \inf_{n \in \mathbb{N}} |\tfrac{1}{2n}| \\ &= 0 \checkmark \end{aligned}$$

b.) Let E_1, E_2 be as described. AFSOC $\nexists$ such a $x_1 \in E_1$, $x_2 \in E_2$.
Let $\{a_n\} \subseteq E_1$ & $\{b_n\} \subseteq E_2$ s.t. $\rho(a_n, b_n) \to 0$. Since E_2 is compact, $\exists$ subseq $\{b_{n_k}\}$ s.t. $b_{n_k} \to b$ for some $b \in E_2$. Then:

$$|b - a_{n_k}| = |b - b_{n_k} + b_{n_k} - a_{n_k}| \leq \underbrace{|b - b_{n_k}|}_{\to 0} + \underbrace{|b_{n_k} - a_{n_k}|}_{\to 0} \to 0$$

additionally, since E_1 is closed and b is a limit pt of $\{a_{n_k}\} \subseteq E_1$, we have that $b \in E_1$. But this contradicts E_1 & E_2 being disjoint $\lightning$
So there must exist $x_1 \in E_1$ + $x_2 \in E_2$ as described in the claim.

4. Suppose that $\{f_k\}$ and $\{g_k\}$ are two sequences of functions in $L^2([0,1])$. Suppose that

$$\|f_k\|_2 \le 1 \quad \text{for all } k\,,$$

where the L^p norm $\|\cdot\|_p$, $1 \le p \le \infty$, is defined with respect to the standard Lebesgue measure μ on $[0,1]$. Suppose further that there exist $f, g \in L^2([0,1])$ such that $f_k(x) \to f(x)$ for $a.e.$ $x \in [0,1]$, and that $g_k \to g$ in $L^2([0,1])$. Prove that $f_k g_k \to fg$ in $L^1([0,1])$.

6.7 Proposition. *For $1 \le p < \infty$, the set of simple functions $f = \sum_1^n a_j \chi_{E_j}$, where $\mu(E_j) < \infty$ for all j, is dense in L^p.*

Proof. Clearly such functions are in L^p. If $f \in L^p$, choose a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ a.e. and $|f_n| \le |f|$ according to Theorem 2.10. Then $f_n \in L^p$ and $|f_n - f|^p \le 2^p|f|^p \in L^1$, so by the dominated convergence theorem, $\|f_n - f\|_p \to 0$. Moreover, if $f_n = \sum a_j \chi_{E_j}$ where the E_j are disjoint and the a_j are nonzero, we must have $\mu(E_j) < \infty$ since $\sum |a_j|^p \mu(E_j) = \int |f_n|^p < \infty$. ∎

conv in L^1 $\overset{\text{WTS:}}{\Rightarrow}$ $\int |f_k g_k|\, d\mu \to \int |fg|\, d\mu$

a.e. x $\Rightarrow |f_k(x) - f(x)| < \varepsilon \quad \forall k \ge N$ for a given x

↳ pointwise on measurable sets

conv in $L^2 \Rightarrow (\int |g_k|^2 d\mu)^{1/2} \to (\int |g|^2 d\mu)^{1/2}$

WTS: $\|f_k g_k - fg\|_1 \to 0$ $\qquad |f+g|^p \le 2^p(|f|^p + |g|^p)$

$\int |\underbrace{f_k g_k - fg}_{\le |f_k g_k| - |fg|}|\, d\mu$

$\{f_k\}, \{g_k\} \subset L^2([0,1])$ $\qquad f, g \in L^2([0,1])$

$\|f_k\|_2 \le 1 \quad \forall k$ $\qquad L^p$ norm def wrt Leb measure

$\{f_k\} \to f$ a.e.

$\{g_k\} \to g$ in $L^2([0,1])$ $\qquad$ WTS: $\{f_k g_k\} \to fg$ in $L^1([0,1])$

Holder: $\|fg\|_1 \le \|f\|_2 \|g\|_2 < \infty \cdot \infty$

$\Rightarrow fg \in L^1([0,1])$

5. Let f be a real-valued function on $[0,\infty)$, such that $f \in L^2([0,\infty))$. (Here the L^p spaces are defined with respect to the standard Lebesgue measure on $[0,\infty)$.) Define $F : [0,\infty) \to \mathbb{R}$ by letting $F(x) = \int_0^x f(t)\,dt$ for $x \geq 0$. Assume that $F \in L^1([0,\infty))$.

(a) Prove that $F(x)$ goes to zero as $x \to \infty$.

(b) Replace the assumption that $f \in L^2([0,\infty))$ with the assumption that $f \in L^1([0,\infty))$, and prove that also with this single change on the conditions, $F(x)$ goes to zero as $x \to \infty$.

Note: Parts (a) and (b) are independent of one another and may be solved in either order, though one can also prove both at once, use one to prove the other, etc.

$f \in L^2([0,\infty)) \Rightarrow \left(\int_0^\infty |f(t)|^2 dt\right)^{1/2} < \infty$

$\Rightarrow \int_0^\infty |f(t)|^2 dt < \infty$

$F \in L^1([0,\infty)) \Rightarrow \int_0^\infty |F(x)|\,dx < \infty$

$\int_0^\infty \left|\int_0^x f(t)\,dt\right| dx$

$f \in L^1([0,\infty)) \Rightarrow \int_0^\infty |f(t)|\,dt < \infty$

$F \in L^2([0,\infty)) \Rightarrow \int_0^\infty |F(x)|\,dx < \infty$

$\int_0^\infty \left|\int_0^x f(t)\,dt\right| dx$

$F(x) \to 0 \Rightarrow |F(x) - 0| \; (< \varepsilon)$

$|F(x) - 0| = \left|\int_0^x f(t)\,dt\right|$

AFSOC $F(x) \not\to 0$. Then $\exists M > 0$ and a seq. $\{x_n\} \to \infty$ s.t. $|F(x_n)| \geq M$ $\forall n$ ($\geq N$, but we just redefine the seq)

★ $f \in L^2 \Rightarrow F(x) = \int_0^x f(t)\,dt$ is uni cont (cr: Riley)

By uni cont, whenever for $\varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $|x-y| < \delta$, $|F(x) - F(y)| < \varepsilon$.

Let $\varepsilon = \frac{M}{2}$, then

$\int_{x_n}^{x_n+\delta} |F(t)|\,dt = \int_{x_n}^{x_n+\delta} |F(t) - F(x_n) + F(x_n)|\,dt$

$\geq \int_{x_n}^{x_n+\delta} \big|\,|F(t) - F(x_n)| - |F(x_n)|\,\big|\,dt$ rev. tri. $|a-b| \geq ||a| - |b||$

$t \geq x_n$ $\quad t \in [x_n, x_n+\delta]$ $\quad F(t) - F(x_n) = \int_0^t - \int_0^{x_n} = \int_{x_n}^t$ $\quad t < x_n + \delta$?

$= \int_{x_n}^{x_n+\delta} \underbrace{|F(x_n)|}_{\geq M} - \underbrace{|F(t) - F(x_n)|}_{> M/2}$

$> \int_{x_n}^{x_n+\delta} M - M/2 \; dt = \frac{M}{2} \cdot \delta$

assume each $[x_n, x_n+\delta)$ disjt by construction of our seq, so

$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \int_0^x = \ldots \int_{x_n}^{x_n+\delta} + \int_{x_{n+1}}^{x_{n+1}+\delta} + \ldots = \infty \left(\frac{M}{2} \cdot \delta\right) = \infty$

contradicts $F \in L^1$, so $F(x) \to 0$ necessarily

b.) $f \in L^1 \Rightarrow F$ is also cont on $[0, \infty)$

$\Rightarrow F$ is Riemann integrable

$\Rightarrow F(x) \to 0$ as $x \to \infty$

SPRING 2015

1. In this problem $(X, \mathcal{M}, \mu)$ denotes an arbitrary measure space.
A) State the monotone convergence theorem.
B) Prove that if $f_n : X \to [0, \infty)$ is measurable and non-negative for each positive integer n, then

Fatou

$$\int_X (\liminf_{n\to\infty} f_n)\, d\mu \;\le\; \liminf_{n\to\infty} \int_X f_n\, d\mu. \tag{1}$$

C) Give an example showing that (1) can be false if some of the functions f_n take negative values.

liminf = sup of infs

a.) $\{f_n\} \subseteq L^+$ & $f_n(x) \le f_{n+1}(x)$ $\forall x, n$ then $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n$

b.) Let $g_k = \inf_{n \ge k} f_n$. Then $\{g_k\}$ is monotone, and $\{g_k\} \subseteq L^+$ since $\{f_n\} \subseteq L^+$. Then by MCT:

$\int \lim g_k = \lim \int g_k$

$\int \lim \inf f_n = \lim \int \inf f_n$

$\int \liminf f_n \le \lim\inf \int f_n$ ✓

c.) $f_n = -\chi_{[0, 1/n]}$

still monotone

$\int \liminf f_n = \int 0 = 0$

$\liminf \int f_n = -1$

$0 \nleq -1$ so MCT fails

2. Let f and f_k, $k = 1, 2, \ldots$, be Lebesgue measurable functions on $[0,1]$ such that $f_k \to f$ almost everywhere in $[0,1]$. Suppose that $M := \sup_k \|f_k\|_\infty < \infty$. Show that for every $g \in L^p([0,1])$, $1 \le p \le \infty$,

$$\lim_{k\to\infty} \int_0^1 f_k g \, dx = \int_0^1 f g \, dx .$$

DCT

smallest value bounding $|f(x)|$

$\|f_k\|_\infty = \inf\{a \ge 0 \mid \mu(\{x \mid |f(x)| > a\}) = 0\}$

all $\|f_k\|_\infty < \infty$ since $\sup < \infty$

greatest smallest bound on $f_k(x)$ is M

Holder: $\mu(X) = \mu([0,1]) = 1 < \infty$

for $p < q$, $\|f\|_p \le \|f\|_q$

Then $\|g\|_1 \le \|g\|_\infty$

↳ recall $g \in L^p$ $\forall p \in [1, \infty]$

$g \in L^1$, so $\|g\|_{L^1} < \infty$

$|f_k g| = |f_k| \cdot |g| \le M|g| \in L^1$ since $g \in L^1$

as $f_k g \in L^1$ and $|f_k g| \le M|g|$

By DCT

$\lim \int f_k g = \int \lim f_k g = \int fg$

↳ RILEY'S

SOLUTION:

$\lim_{k\to\infty} \|f_k g\|_{L^1([0,1])} \overset{?}{=} \|fg\|_{L^1([0,1])}$

$\|f_k g\|_{L^1([0,1])} \le \underbrace{\|f_k\|_q}_{\le M} \|g\|_p \qquad \frac{1}{p} + \frac{1}{q} = 1 \qquad$ Holder's

$\le M \cdot \|g\|_p = M \cdot \left(\int g^p \, dx\right)^{1/p}$

$\|f_k g\|_{L^1([0,1])} \le \underbrace{\|f_k\|_\infty}_{\le M} \|g\|_1 \le M \cdot \|g\|_1 = M \int_0^1 g \, dx$

$\lim_{k\to\infty} \|f_k g\|_{L^1([0,1])} \le \lim_{k\to\infty} \|f_k\|_\infty \|g\|_1$

$\lim_{k\to\infty} \int_0^1 f_k g \, dx = \int_0^1 fg \, dx$

$\int_0^1 f_k g \, dx - \int_0^1 fg \, dx = \int_0^1 f_k g - fg \, dx = \int_0^1 g(f_k - f) \, dx$

WTS: $(\to 0)$

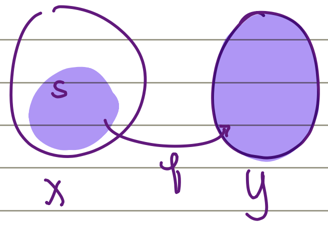
3. Let (X, d_X) be a metric space and let (Y, d_Y) be a complete metric space. Let S be a dense subset of X and let $f: S \to Y$ be an L-Lipschitz map, meaning that $d_Y(f(x), f(y)) \le L d_X(x, y)$ for all $x, y \in S$. Show there is a unique L-Lipschitz map $\bar{f}: X \to Y$ such that $\bar{f}|_S = f$ (that is, $\bar{f}$ is an extension of f).

const?

every Cauchy seq converges

S dense ⇒ $cl(S) = X$

any conv seq in S has lim pt in X



Let $\{x_n\} \subseteq S$ s.t. $\{x_n\} \to x$. Since S is dense, $x \in X$. Then define $\bar{f}(x) = \lim f(x_n)$. Since $\{x_n\}$ converges in X, $\{f(x_n)\}$ converges in Y as f is cont. $\{f(x_n)\}$ being convergent implies that the seq is Cauchy, and since Y is complete we know that $\lim f(x_n)$ both exists and is in Y.

CR: RILEY

Check that $\bar{f}$ is well-defined. Let $\{x_n\}, \{y_n\} \subseteq S$ s.t. both seq converge to x. Then both $\lim f(x_n)$ + $\lim f(y_n)$ exist by the argument above. Considering the seq $x_1, y_1, x_2, y_2, \dots$ we see that the interlaced seq also converges to x, so $f(x_1), f(y_1), f(x_2), \dots$ converges to some $y \in Y$. Since $\{f(x_n)\}$ and $\{f(y_n)\}$ are subseq of the interlaced seq, they must have the same limit as the interlaced seq. Hence $\lim f(x_n) = \lim f(y_n)$, so $\bar{f}(x)$ is well-def.

Check that $\bar{f}|_S = f$. If $x \in S$, then WTS: $\bar{f}(x) = f(x)$.
Let $\{x_n\}$ be the const seq so $x_n = x$ $\forall n$.
Then $\bar{f}(x) = \lim f(x_n) = \lim f(x) = f(x)$ ✓

Check that $\bar{f}$ is L-Lipschitz, so $d_Y(\bar{f}(x), \bar{f}(y)) \le L d_X(x, y)$. Let $\{x_n\} \to x$, $\{y_n\} \to y$.

$$d_Y(\bar{f}(x), \bar{f}(y)) = d_Y(\lim f(x_n), \lim f(y_n)) = \lim d_Y(f(x_n), f(y_n))$$

f is L-Lipschitz, so $d_Y(f(x_n), f(y_n)) \le L d_X(x_n, y_n)$ $\forall n$

$$\begin{aligned} d_Y(\bar{f}(x), \bar{f}(y)) &= \lim d_Y(f(x_n), f(y_n)) \\ &\le \lim L d_X(x_n, y_n) \\ &= L d_X(\lim x_n, \lim y_n) \\ &= L d_X(x, y) \end{aligned}$$

$\Longrightarrow d_Y(\bar{f}(x), \bar{f}(y)) \le L d_X(x, y)$ ✓

Check that $\bar{f}$ is unique. AFSOC $\exists\, \bar{g}$ another extension s.t. $\bar{f} \ne \bar{g}$. Then $\exists\, x \in X$ where $\bar{f}(x) \ne \bar{g}(x)$. $\Rightarrow 0 < d_Y(\bar{f}(x), \bar{g}(x))$
Let $y \in S$ s.t. $d_X(x, y) < \varepsilon$. Then $d_Y(\bar{f}(x), f(y)) < L\varepsilon$. Note $\bar{f}(y) = f(y)$ as $y \in S$.
Similarly, $d_Y(\bar{g}(x), g(y)) < L\varepsilon$ and $\bar{g}(y) = g(y)$ //
(both $\bar{f}$ and $\bar{g}$ are L-Lipschitz + satisfy restriction to S)

$$\begin{aligned} \text{Then } d_Y(\bar{f}(x), \bar{g}(x)) &\le d_Y(\bar{f}(x), \bar{f}(y)) + d_Y(\bar{f}(y), \bar{g}(x)) \\ \text{metric} &= d_Y(\bar{f}(x), f(y)) + d_Y(g(y), \bar{g}(x)) \\ &< L\varepsilon + L\varepsilon \\ &= 2L\varepsilon \end{aligned}$$

$L^p \supseteq L^q \quad \forall p < q$

So $d_Y(\bar{f}(x), \bar{g}(x)) < 2L\varepsilon$. But this is true for any arbitrarily small ε, so $d_Y(\bar{f}(x), \bar{g}(x)) \to 0$, and hence $\bar{f}(x) = \bar{g}(x)$. This is true $\forall x \in X$, so $\bar{f} \equiv \bar{g}$ and thus $\bar{f}$ is unique.

4. Let H be a Hilbert space endowed with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Recall that a sequence $\{u_k\}_{k\geq 1}$ in H is said to *converge weakly* to a limit $u \in H$ if for all $v \in H$, $\lim_{k\to\infty}(v, u_k - u) = 0$. In contrast, $\{u_k\}_{k\geq 1}$ *converges strongly* to $u \in H$ if $\lim_{k\to\infty}\|u_k - u\| = 0$.

Suppose that a sequence $\{u_k\}_{k\geq 1}$ in H converges weakly to $u \in H$, and that furthermore $\lim_{k\to\infty}\|u_k\|_2 = \|u\|_2$. Show that $\{u_k\}_{k\geq 1}$ in fact converges strongly to u.

Given $u_k \to u$ weakly

$\forall v,\ \lim (v, u_k - u) = 0$

WTS: $u_k \to u$ strongly

$\lim \|u_k - u\| = 0$

$$\|u_k - u\|^2 = (u_k - u, u_k - u)$$
$$= (u_k, u_k - u) - (u, u_k - u)$$
$$= \underbrace{(u_k, u_k)}_{=\|u_k\|^2} - (u_k, u) - (u, u_k) + \underbrace{(u, u)}_{=\|u\|^2}$$
$$= \|u_k\|^2 + \|u\|^2 - \left((u_k, u) + (u, u_k)\right)$$

Using weak conv, $\lim(v, u_k - u) = 0$

let $v = u$:

$$0 = \lim(u, u_k - u) = \lim\left((u, u_k) - (u, u)\right) = \lim\left((u, u_k) - \|u\|^2\right)$$

let $v = u_k$:

$$0 = \lim(u_k, u_k - u) = \lim\left((u_k, u_k) - (u_k, u)\right) = \lim\left(\|u_k\|^2 - (u_k, u)\right)$$

$$\lim \|u_k - u\|^2 = \lim\left(\|u_k\|^2 + \|u\|^2 - (u_k, u) - (u, u_k)\right)$$
$$= \lim\left(\|u_k\|^2 - (u_k, u)\right) + \lim\left(\|u\|^2 - (u, u_k)\right)$$
$$= 0 + 0 = 0$$

$\Rightarrow \lim \|u_k - u\|^2 = 0$

$\Rightarrow \lim \|u_k - u\| = 0$ ✓

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $f_n, g_n, h_n \in L^1(X, \mathcal{M}, \mu)$, $n \geq 1$, satisfy the inequalities

$$f_n(x) \leq g_n(x) \leq h_n(x), \quad \text{for all } n \geq 1 \text{ and } x \in X.$$

Let $f(x)$, $g(x)$, and $h(x)$ be functions such that

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \lim_{n\to\infty} g_n(x) = g(x), \quad \text{and} \quad \lim_{n\to\infty} h_n(x) = h(x),$$

for almost all $x \in X$. Furthermore assume that $f(x), h(x) \in L^1(X)$ with

$$\lim_{n\to\infty} \int_X f_n(x)\,d\mu = \int_X f(x)\,d\mu \quad \text{and} \quad \lim_{n\to\infty} \int_X h_n(x)\,d\mu = \int_X h(x)\,d\mu.$$

Show that

$$\lim_{n\to\infty} \int_X g_n(x)\,d\mu = \int_X g(x)\,d\mu.$$

(HINT: Look at $h_n - g_n$ and $g_n - f_n$.)

$f_n \leq g_n \leq h_n \quad \in L^1$

even $j_n \to j$ for a.e. $x \in X$

$|h_n - g_n| \geq 0$

$|g_n - f_n| \geq 0$

$f_n \leq g_n \leq h_n$

$0 \leq \underbrace{g_n - f_n}_{\text{pos.}} \leq \underbrace{h_n - f_n}_{\text{pos.}} \quad \Rightarrow \quad |g_n - f_n| \leq \underbrace{h_n - f_n}_{\in L^1}$

$\{g_n - f_n\} \subset L^1$ w/ $g_n - f_n \to g - f$ a.e.

$|g_n - f_n| \leq h_n - f_n$ w/ $h_n - f_n$ nonneg.

By DCT: $\lim \int g_n - f_n = \int g - f$

$\lim \int g_n - f_n = \int g - \int f$

$\lim \int g_n - \lim \int f_n = \int g - f$

$\lim \int g_n - f = \int g - \int f$

$\lim \int g_n = \int g$ ✓

FALL 2015

1. a) Let m denote Lebesgue measure on $\mathbb{R}$. Prove that the subset A of $L^1(m)$ defined by $A := \{f \in L^1(m) : \int_{\mathbb{R}} |f| dm \leq 1\}$ is closed under pointwise convergence.

b) Prove that the set $B := \{f \in L^1(m) : \int_{\mathbb{R}} |f| dm \geq 1\}$ is not closed under pointwise convergence.

a.) $f \in L^1 \Rightarrow |f| \in L^+$, let $g_n = |f_n|$

By Fatou: $\int \liminf g_n \leq \underbrace{\liminf \int g_n}_{=1}$

For conv seq. $\{g_n\} \to g$, have

$\int \liminf g_n \leq 1$

$\int g \leq 1$ so $g \in A$ & A is closed under ptwise conv.

b.) Let $f_n = n \chi_{[0,1/n]}$

Then $\int |f_n| = \int n \chi_{[0,1/n]} = n \cdot \frac{1}{n} = 1 \geq 1$ ✓

But $\int \lim |f_n| = \int \lim n \chi_{[0,1/n]} = \int \chi_{\{0\}} \lim_{n\to\infty} n = 0$

So $0 \ngeq 1$ and B is not closed

2. a) For α a real number and $\alpha > -1$, prove that $\int_0^\infty x^\alpha e^{-x} dm < \infty$, where m denotes Lebesgue measure on $\mathbb{R}$.

b) For $\alpha > -1$ and k a positive integer, prove that

$$\lim_{k\to\infty} \int_0^k x^\alpha \left(1 - \frac{x}{k}\right)^k dm = \int_0^\infty x^\alpha e^{-x} dm.$$

↳ $x \geq 0$

a.) $\int_0^\infty \underbrace{x^\alpha}_{u} \underbrace{e^{-x} dm}_{dv} = ?$ $\Rightarrow v = -e^{-x}$ int by parts $\left(\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \right.$

$\int_0^\infty x^\alpha e^{-x} dm = x^\alpha \cdot -e^{-x} \Big|_0^\infty - \int_0 \alpha x^{\alpha-1} \cdot -e^{-x} dm$

$= -\Big(\underbrace{\frac{\infty^\alpha}{e^\infty}}_{=0} - \underbrace{\frac{0^\alpha}{e^0}}_{=0}\Big) + \alpha \int_0^\infty x^{\alpha-1} e^{-x} dm = \alpha \int_0^\infty x^{\alpha-1} e^{-x} dm$

Can induct on α since $\int_0^\infty x^\alpha e^{-x} dm = \alpha \int_0^\infty x^{\alpha-1} e^{-x} dm$

Check each integral $< \infty$, then!

Base case: $\alpha \in [-1, 0]$ then increment each interval by 1 ($[1,2], [2,3], \ldots$)

$\int_0^\infty x^\alpha e^{-x} dm = \int_0^1 x^\alpha e^{-x} dm + \int_1^\infty x^\alpha e^{-x} dm$

↳ $x^\alpha \leq 1$ for $\alpha \in [-1, 0]$ when $x \geq 1$

↳ $e^{-x} \leq 1$ for $x \in [0,1]$

$\leq \int_0^1 x^\alpha \cdot 1\, dm + \int_1^\infty 1 \cdot e^{-x} dm$

$= \underbrace{\frac{1}{\alpha+1}}_{\leq 1} + \underbrace{\frac{1}{e}}_{<1} < \infty$ ✓

$\int_1^\infty 1 \cdot e^{-x} dm = -e^{-x}\Big|_1^\infty = -(e^{-\infty} - e^{-1}) = -(0 - e^{-1}) = \frac{1}{e}$

$\int_0^1 x^\alpha \cdot 1\, dm = \frac{1}{\alpha+1} x^{\alpha+1}\Big|_0^1 = \frac{1}{\alpha+1}(1^{\alpha+1} - 0^{\alpha+1}) = \frac{1}{\alpha+1}(1-0) = \frac{1}{\alpha+1}$

b.) $\lim_{k\to\infty} \int_0^k x^\alpha \left(1 - \frac{x}{k}\right)^k dm = \lim_{k\to\infty} \int_0^\infty x^\alpha \left(1 - \frac{x}{k}\right)^k \chi_{[0,k]}\, dm$ → don't know if monotone, but we do know $f \in L^1$

want to pull limit inside, so try ~~MCT~~ or DCT

$f_k(x) = x^\alpha \left(1 - \frac{x}{k}\right)^k \chi_{[0,k]}$

Need $f_k(x) \in L^1$ + $|f_k| \leq g$ for some $g \in L^1$

also need $f_k(x) \to \underbrace{x^\alpha e^{-x}}_{=f}$ a.e.

3. Let (K,d) be a compact metric space which is *well-tied*, which means that for every $\epsilon > 0$, $x \in K$, and $y \in K$, there is a **finite** sequence of points

$$x = x_1,\ x_2,\ \dots,\ x_n = y \ \text{ in } K \qquad \text{such that} \qquad d(x_i, x_{i+1}) \le \epsilon$$

(n might depend on x and on y).

a) Assume that K can be written as the disjoint union $K = F_1 \cup F_2$, where F_1 and F_2 are both **closed** and nonempty subsets of K. Prove that $d(F_1, F_2) = \inf_{x \in F_1, y \in F_2} d(x,y) > 0$.

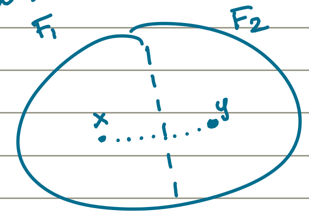
b) Show that K is connected.

Hint for part b: Prove that the compact metric space K cannot be written as a disjoint union of two closed, nonempty subsets.

or inf.

This equality is simply the definition of dist. btwn sets in a metric space

a.)



For any two $x, y \in K$,
$d(x,y) \le n \cdot \varepsilon$ since $\exists$ seq. $\{x_n\}$ w/ $d(x_i, x_{i+1}) \le \varepsilon$
and by def of metric, $d(x,y) \le \sum_{i=1}^{n} d(x_i, x_{i+1})$
$\le \sum_{i=1}^{n} \varepsilon = n\varepsilon$

Clearly by def of dist $d(F_1, F_2)$ btwn sets
$d(F_1, F_2) \le \inf_{x \in F_1, y \in F_2} d(x,y)$

Closed = cont all limit pts

Disjoint

↳ why $d(F_1, F_2) > 0$

Take some $x \in F_1$. For any $y \in F_2$, construct infinite seq. $\{x_n\}$ btwn them according to the given condition but s.t. $x_j = x_n$ $\forall j > n$. Then $\{x_n\} \to y$. Since F_1 is closed, it contains all limit pts of seq contained F_1, so $y \in F_1$. However, this contradicts F_1 & F_2 disjt as both cont. y

Then this cannot be possible. Conclude that the entire seq. $\{x_n\} \not\subset F_1$, i.e. $\exists$ some smallest $i < n$ s.t. $x_i \in \{x_n\}$ satisfies $x_i \notin F_1$. Then $d(x_{i-1}, x_i) > 0$ since $x_{i-1} \in F_1$ and $x_i \notin F_1$. Note x_i may not be in F_2, but we must "exit" F_1 to obtain $x_n = y \in F_2$ in our seq. These conditions hold for any $x \in F_1$, $y \in F_2$, so we have that $\inf_{x \in F_1, y \in F_2} d(x,y) > 0$.

b.) WTS: compact metric space cannot be written as a disjt union of two closed, nonempty sets

By pt (a.), if you could write $K = F_1 \cup F_2$ w/ $F_1 \cap F_2 = \emptyset$, then $d(F_1, F_2) > 0$.

Let $d(F_1, F_2) = \varepsilon_0 > 0$. Then $\exists$ some pt z at a distance $\varepsilon_0/2$ from both F_1 & F_2. But K is compact, so must contain z. So one of F_1 or F_2 must contain z, which contradicts part (a.) as we can't have $d(F_1, F_2) > 0$.

Hence K cannot be disconnected, so is connected.

could use K cpt $\Rightarrow$ closed (metric space)
$\Rightarrow$ cont. all limit pts

construct seq. converging to z, voila!

4. a) Let $[a,b]$ be a closed, bounded interval and $f:[a,b]\to\mathbb{R}$. Give an "epsilon-delta definition" of what it means for f to be "absolutely continuous on $[a,b]$."

f continuous $\Rightarrow$ for any $\varepsilon>0$, $\exists\delta>0$ s.t. when $|f(x)-f(y)|<\varepsilon$, $|x-y|<\delta$

abs cont $\Rightarrow$ for any $\varepsilon>0$, $\exists\delta>0$ s.t.

for a finite set of disjt intervals $\{(a_i,b_i)\mid i\in[1,n]\}$

$\sum_1^n (b_i-a_i)<\delta \Rightarrow \sum_1^n |f(b_i)-f(a_i)|<\varepsilon$

} pg. 105 Folland

b) Assume now that $f:[0,1]\to\mathbb{R}$ has the property that for every ε with $0<\varepsilon<1$, the restriction of f to the closed interval $[\varepsilon,1]$ is absolutely continuous. Assume also that there exists some $p>2$ such that

$$\int_0^1 x|f'(x)|^p\,dm<\infty,$$

where m denotes Lebesgue measure. Prove that $\lim_{x\to0^+} f(x)$ exists and is finite.

$\hookrightarrow \lim_{x\to0^+} = \lim_{\varepsilon\to0}$

$\forall\varepsilon>0$, $f|_{[\varepsilon,1]}$ abs cont

$\Rightarrow f$ diff'able on $[a,b]=[\varepsilon,1]$

$f'\in L^1([a,b],m)$ + $f(x)=f(a)+\int_a^x f'(t)\,dt$ } FTC

$f'\in L^1([\varepsilon,1],m)$ $\quad f(x)=f(\varepsilon)+\int_\varepsilon^x f'(t)\,dt$

Note: File has sols on Rutgers site

Q2 similar to this!

also have $\int_0^1 x|f'(x)|^p\,dm<\infty$

Given info about $\int_0^1$, so let $a=0$ and $x=1$

$a=\varepsilon$, then send to 0

$f(1)=f(\varepsilon)+\int_\varepsilon^1 f'(t)\,dt$

$f(\varepsilon)=f(1)-\underbrace{\int_\varepsilon^1 f'(t)\,dt}_{\text{look at this guy}}$

$\int_\varepsilon^1 |f'(t)|\,dt = \int_\varepsilon^1 |f'(t)\cdot x^{1/p}\cdot x^{-1/p}|\,dt$

$= \int_\varepsilon^1 |(f'(t))^p x|^{1/p}\cdot|x^{-1/p}|\,dt$ $\qquad x\in[0,1]$ so $|x|=x$

$= \int_\varepsilon^1 x^{1/p}|f'(t)^p|^{1/p}\,x^{-1/p}\,dt$

$= \int_\varepsilon^1 x^{1/p}|f'(t)|\,x^{-1/p}\,dt$ $\qquad = \|x^{1/p}f'(t)\,x^{-1/p}\|_{L^1}$

$1=\frac{1}{p}+\frac{p-1}{p}$ so by Holder:

$\le \|x^{1/p}f'(t)\|_{L^p} + \|x^{-1/p}\|_{L^{\frac{p-1}{p}}}$

$= \left(\int_\varepsilon^1 |x^{1/p}f'(t)|^p\,dt\right)^{1/p} + \left(\int_\varepsilon^1 |x^{-1/p}|^{\frac{p-1}{p}}\,dt\right)^{\frac{p}{p-1}}$

$= \underbrace{\underbrace{\left(\int_\varepsilon^1 x|f'(t)|^p\,dt\right)^{1/p}}_{<\infty} + \underbrace{\left(\int_\varepsilon^1 x^{\frac{1-p}{p^2}}\,dt\right)^{p/p-1}}_{<\infty}}_{<\infty}$

$\hookrightarrow \frac{1-p}{p^2}<1$ + intable

$\Rightarrow \int_\varepsilon^1 |f'(t)|\,dt<\infty$

$\lim_{x\to0^+} f(x) = \lim_{\varepsilon\to0} f(\varepsilon) = \lim_{\varepsilon\to0} f(1)-\int_\varepsilon^1 f'(t)\,dt$

$= \underbrace{f(1)}_{<\infty} - \underbrace{\int_0^1 f'(t)\,dt}_{<\infty} < \infty$ ✓

$\hookrightarrow$ continuous + L^1 on $[\varepsilon,1]$

*

5. a) For $t \in [0,1]$ and $n \geq 0$, let $u_n(t)$ be the sequence of continuous functions defined by $u_0(t) = 0$, $\forall t \in [0,1]$, and by the recursion formula

$$u_{n+1}(t) = u_n(t) + \tfrac{1}{2}(t - u_n(t)^2).$$

Prove that $u_{n+1}(t) \geq u_n(t)$ and $0 \leq u_n(t) \leq \sqrt{t}$, $\forall t \in [0,1]$ and $\forall n \geq 0$.

b) Prove that the sequence of continuous functions $u_n(t)$ converges uniformly to a continuous function $f(t)$. What is $f(t)$?

$u_0(t) = 0$

$u_1(t) = 0 + \frac{1}{2}(t - 0^2) = \frac{1}{2}t$

$u_2(t) = \frac{1}{2}t + \frac{1}{2}(t - (\frac{1}{2}t)^2) = \frac{1}{2}t + \frac{1}{2}(t - \frac{1}{4}t^2) = t - \frac{1}{8}t^2$

$u_{n+1}(t) = u_n(t) + \frac{1}{2}(t - u_n(t)^2)$

WTS: $u_{n+1} \geq u_n$ $\forall t$ AND $0 \leq u_n(t) \leq \sqrt{t}$ $\forall t, n$

$0 \leq u_n(t) \leq \sqrt{t}$ by induction:

Base: $0 \leq u_0(t) \leq \sqrt{t}$

$0 \leq 0 \leq \sqrt{t}$ $\forall t \in [0,1] \rightsquigarrow \sqrt{t} \in [0,1]$ ✓

IH: $0 \leq u_n(t) \leq \sqrt{t}$

IS: consider $u_{n+1}(t)$

$u_{n+1}(t) = u_n(t) + \frac{1}{2}(t - u_n(t)^2)$

$\underbrace{u_n(t)}_{\geq 0} + \frac{1}{2}(\underbrace{t}_{\geq 0} - \underbrace{u_n(t)^2}_{\geq 0}) \geq 0 + \frac{1}{2}(0 - 0^2) = 0$ ✓

$\underbrace{u_n(t)}_{\leq \sqrt{t}} + \frac{1}{2}(t - \underbrace{u_n(t)^2}_{\leq (\sqrt{t})^2 = t}) \leq \sqrt{t} + \frac{1}{2}(t - t) = \sqrt{t} + 0 = \sqrt{t}$ ✓

$\Rightarrow 0 \leq u_{n+1}(t) \leq \sqrt{t}$

Given this:

$u_{n+1}(t) = u_n(t) + \frac{1}{2}(t - u_n(t)^2)$

$u_{n+1}(t) - u_n(t) = \frac{1}{2}(t - u_n(t)^2)$

WTS: $(\;) \geq 0$

$0 \leq u_n(t) \leq \sqrt{t} \Rightarrow 0 \leq u_n(t)^2 \leq t$

$0 \geq -u_n(t)^2 \geq -t$

$t \geq t - u_n(t)^2 \geq 0$

$\Rightarrow (t - u_n(t)^2) \geq 0$ ✓

b.) Dini's Thm: a monotone seq. of continuous funcs which conv. ptwise to a cont. limit conv. uniformley to the limit on cpt. sets

From part (a.):

- $\{u_n\}$ monotone $\quad u_n \leq u_{n+1}$ $\forall n$
- $u_n(t) \in [0,1]$ $\forall t$ $\quad 0 \leq u_n(t) \leq \sqrt{t} \leq 1$ $\quad \forall t \in [0,1]$

Then, suffices to identify ptwise limit + show cont...

$u_n(t) \leq \sqrt{t} \leq 1 \Rightarrow$ pointwise bounded $\Rightarrow u(t) \leq 1$ $\forall t \in [0,1]$

$\lim_{n \to \infty} u_n(t) = u(t)$ $\forall t \in [0,1]$ (pointwise limit)

at the limit, have $u_\infty = u_{\infty+1}$ so under recursive definition:

$u_{\infty+1}(t) = u_\infty(t) + \frac{1}{2}(t - u_\infty(t)^2)$

$=$

$0 = \frac{1}{2}(t - u_\infty(t)^2)$

$0 = t - u(t)^2$

$u(t)^2 = t \Rightarrow \boxed{u(t) = \sqrt{t}}$ This is cont. on $[0,1]$, so Dini's Thm gives us uni conv.

SPRING 2016

1. A subset A of $\mathbb{R}^n$ is said to be *path-connected* if, given any two points $x_0, y_0 \in A$, there exists a continuous path $\phi : [0,1] \longrightarrow A$ such that $\phi(0) = x_0$ and $\phi(1) = y_0$.

a) Prove that if $A \subset \mathbb{R}^n$ is non-empty and path-connected, then A is connected. $X \neq A \cup B$ for A, B disjt, nonempty, open

b) Suppose now that A is an open subset of $\mathbb{R}^n$. For $x \in A$, let C_x be the set of points z in A for which there is a continuous path in A from x to z. Prove that C_x is open in A. (Hint: use the fact that every ball in $\mathbb{R}^n$ is path-connected, and use composition of paths.)

c) Continuing with the assumptions of part b), prove that for any two points $x, y \in A$, either $C_x = C_y$ or $C_x \cap C_y = \emptyset$.

d) Continuing with the assumptions of parts b) and c), prove that if A is connected, then A is also path-connected. (Hint: use the fact that A can be written as $\cup_{x \in A} C_x$.)

$[0,1]$ connected
so ϕ cont has to preserve

a.) A is nonempty, path connected.
AFSOC $\exists X, Y$ open, nonempty, disjt s.t. $A = X \cup Y$.
Let $x \in X$, $y \in Y$ w/ ϕ the path between them. ($\phi : [0,1] \to A$)
Note ϕ is continuous by def, so ϕ preserves openness.
Also note that $[0,1]$ is connected.
$\phi^{-1}(A) = \phi^{-1}(X \cup Y) = \phi^{-1}(X) \cup \phi^{-1}(Y)$
X, Y open, so $\phi^{-1}(X)$ and $\phi^{-1}(Y)$ open
" disjt, so " disjt
$\phi^{-1}(A) = [0,1] = \phi^{-1}(X) \cup \phi^{-1}(Y)$ where $\phi^{-1}(X), \phi^{-1}(Y)$ disjt, nonempty, open.
Contradicts $[0,1]$ being connected since this is a partition $\lightning$

b.) A open. C_x = set of pts for which there is a connected path to x
WTS: C_x open ($\exists B_\varepsilon(z) \subset C_x \; \forall z \in C_x$)
Take $z \in C_x$. Consider $B_\varepsilon(z)$. WTS: $\subseteq C_x$
$z \in C_x \Rightarrow \exists \phi$ btwn x + z. Every ball in $\mathbb{R}^n$ is path connected, so $\exists \varphi : [0,1] \to B_\varepsilon$ s.t.
$\varphi(0) = z$ + $\varphi(1) = z_0$ for any $z_0 \in B_\varepsilon$.
Consider the path $\psi : [0,1] \to A$ defined by

$$\psi(t) = \begin{cases} \phi(2t) & 0 \le t \le \frac{1}{2} \\ \varphi(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Then $\psi(0) = x$, $\psi(1) = z_0$
$\psi : [0,1] \to A$ is continuous as the composition of 2 cont maps.

$\Rightarrow \exists$ path from x to $z_0 \in B_\varepsilon(z) \; \forall z_0$, hence
$B_\varepsilon(z) \subseteq C_x$ + C_x open. ✓

c.) $x, y \in A \Rightarrow$ either $C_x = C_y$ or $C_x \cap C_y = \emptyset$
assume $C_x \cap C_y \neq \emptyset$. Then $\exists z \in C_x, C_y$. By the argument from part (b) using pathwise composition, $\exists \phi_1$ from x to z and ϕ_2 from z to y. Then $\exists \phi_3$ from y to any $y_0 \in C_y$, and we can form a path between x and y_0 by composing all 3 paths. Then $y_0 \in C_x$ as well. The same is true for any $x_0 \in C_x$ as $\exists \phi_4$ from x_0 to x, and composing ϕ_4, ϕ_1, + ϕ_2 produces a path from x_0 to y and hence $x_0 \in C_y$. Thus, $C_x \subseteq C_y$ + $C_y \subseteq C_x$, so if $C_x \cap C_y \neq \emptyset$, $C_x = C_y$ by double containment.

d.) WTS: connected $\Rightarrow$ path connected.
Given $A = \cup_{x \in A} C_x$. Assume A is connected, WTS: $\exists \phi$ btwn any 2 pts x, y.
aka any 2 C_x, C_y have $C_x \cap C_y \neq \emptyset$
since then $C_x = C_y$ so $\exists \phi$ from x to y.

Assume $C_x \cap C_y = \emptyset$. C_x is open by part (b.). Consider $B = A \setminus C_x$. Note $\forall y \in B$, $y \notin C_x$. Then additionally, $C_y \subseteq B$, otherwise $\exists\, y_0 \in C_y$ s.t. $y_0 \notin B$, hence $y_0 \in C_x$, but then $\exists$ path ϕ from x to y by composing ϕ_1 from x to y_0 and ϕ_2 from y_0 to y (contradicts $y \notin C_x$.) In fact, $B = \bigcup_{y \in B} C_y$ since each $C_y \subseteq B$ and every $y \in B$ is in C_y, which is included in the union def. Since B is a union of open sets, B is also open.
Then C_x, B both open + nonempty, $C_x \cap B = \emptyset$ by construction, +
$C_x \cup B = C_x \cup (A \setminus C_x) = A$
This is a partition of A, which contradicts A being connected.

Hence, $C_x \cap C_y \neq \emptyset$ $\forall x, y$, so $C_x = C_y$ by part (c.) and $\exists\, \phi$ from x to y for any $x, y \in A$. So A is path connected.

2. Let $[a,b]$ be a (bounded) interval of $\mathbb{R}$ and let m be Lebesgue measure. Let M be a positive real number and let $f_1, f_2, \ldots$ be a sequence of measurable functions on $[a,b]$ for which $\int_a^b |f_n|\,dm \le M$ for every n. Assume that $f_n(x) \to f(x)$ as $n \to \infty$ for m-almost every x.

a) State Fatou's lemma.

b) Show that $\int_a^b |f|\,dm \le M$.

c) Suppose that $\|f_n - f\|_1 \to 0$. Prove for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subset [a,b]$ is m-measurable and $m(A) \le \delta$, then $\int_A |f_n|\,dm \le \epsilon$ for all n.

a.) $\{f_n\} \subseteq L^+$ Then $\int \liminf f_n \le \liminf \int f_n$

b.) $f_1, f_2, \ldots$ all msble, so $\{|f_n|\} \subseteq L^+$. and $f_n \to f \Rightarrow |f_n| \to |f|$
Then $\underbrace{{}_a\int^b \liminf |f_n|}_{= \lim f_n = f} \le \underbrace{\liminf {}_a\int^b |f_n|}_{= M}$
$\Rightarrow {}_a\int^b |f| \le M$ ✓

c.) $\|f_n - f\|_1 \to 0$

44. (Lusin's Theorem) If $f : [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \epsilon$ and $f|E$ is continuous. (Use Egoroff's theorem and Theorem 2.26.)

For any $\varepsilon > 0$, use Lusin and f_n, to give compact set $A_n \subset [a,b]$ s.t. $\mu(A_n) < \underbrace{\varepsilon/M}_{= \delta}$. Then ${}_{A_n}\int |f_n|\,dm < \underset{(M \cdot L)}{M} \cdot \varepsilon/M = \varepsilon$ for each n... individually

$\|f_n - f\|_1 \to 0 \Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \ge N$, ${}_a\int^b |f_n - f| < \varepsilon$
For our given ε, let $\{A_n\}$ be the seq of sets selected by Lusin, and let A be the set selected for f itself.
Pick $A^* = A \cap \bigcap_{n=1}^{N} A_n$. Then since each A_n satisfies $m(A_n) < \delta$, monotonicity of measure gives us that $m(A^*) < \delta$.

Then $\forall n < N$: ${}_{A^*}\int |f_n| \le {}_a\int^b |f_n| < \varepsilon$ ✓

and $\forall n \ge N$: ${}_{A^*}\int |f_n| = {}_{A^*}\int |f_n - f + f| \le {}_{A^*}\int |f_n - f| + {}_{A^*}\int |f| < 2\varepsilon$ ✓

$\le {}_a\int^b |f_n - f| < \varepsilon$ by conv in L^1

$\le {}_A\int |f| < \varepsilon$

3. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ satisfy:

(i) for each $x \in [0,1]$, the function $y \mapsto f(x,y)$ is Riemann integrable on $[0,1]$; and

(ii) for each $y \in [0,1]$, the function $x \mapsto f(x,y)$ is Borel measurable.

Show that the function $g(x) := \int_0^1 f(x,y)\,dy$ is Borel measurable.

(Note: In general, f will not be Borel measurable as a function on $[0,1] \times [0,1]$.)

CR: TIM

<u>WTS</u>: $g(x) := \int_0^1 f(x,y)\,dy$ is measurable

Folland pg. 57

By (i.), since $y \mapsto f(x,y)$ is Riemann integrable on $[0,1]$, for any partition $P_n = [y_0 = 0, y_1, \dots, y_n = 1]$ of $[0,1]$, we have

$$g_n(x) = \sum_{k=1}^{n} f(x, y_k)(y_k - y_{k-1})$$

is a finite linear combo of Borel measurable funcs, hence $\{g_n(x)\}$ is a seq. of Borel measurable funcs.

Letting $n \to \infty$, our partition P_n is refined, + so $\lim g_n(x) = \int_0^1 f(x,y)\,dy = g(x)$

Since the pointwise limit of a sequence of Borel measurable funcs is also Borel measurable, $g(x)$ is Borel msable, as desired.

4. a) Define Lebesgue outer measure $m^*(A)$ for subsets A of $\mathbb{R}$.

b) If $B \subset \mathbb{R}$ and α is a positive real number, define $\alpha B = \{\alpha x \mid x \in B\}$. Show that $m^*(\alpha B) = \alpha\, m^*(B)$.

c) Show from the definition of Lebesgue integral that for any positive, Lebesgue measurable function f and positive real number α,

$$\int f(x/\alpha)\, m(dx) = \alpha \int f(x)\, m(dx).$$

a.) $m^*(A) = \inf\{\sum_i m(E_i) \mid \cup_i E_i \supset A\}$

b.) $m^*(\alpha B) = \inf\{\sum_i m(F_i) \mid \cup_i F_i \supset \alpha B\}$ $\quad F = \alpha E$

$= \inf\{\sum_i m(\alpha E_i) \mid \cup_i E_i \supset B\}$

$= \inf\{\alpha \sum_i m(E_i) \mid \cup_i E_i \supset B\}$ $\quad$ since m is a measure

$= \alpha \inf\{\sum_i m(E_i) \mid \text{"}\}$

$= \alpha m^*(B)$ ✓

c.) $\int f(x/\alpha)\, m\,dx = \int f(y)\, m\,d(\alpha y) = \int f(y)\, m\,\alpha\,dy = \alpha \int f(y)\, m\,dy$ ✓

subs $y = x/\alpha$, $\alpha y = x$

$d(\alpha y) = \alpha\,dy$ since Leb. measure

5. Let m denote Lebesgue measure on $\mathbb{R}$ and let m^2 denote Lebesgue measure on $\mathbb{R}^2$. Let $f \in L^1(\mathbb{R})$.

a) Show that $h(x,y) = f(x)f(x+y) \in L^1(\mathbb{R}^2, m^2)$.

b) Show that for Lebesgue almost every y, $x \mapsto f(x)f(x+y)$ defines a function in $L^1(\mathbb{R})$.

c) Give an example of a function $g \in L^1(\mathbb{R}, m)$ for which $x \to g(x)g(x+y)$ is **not** in $L^1(\mathbb{R})$ for at least one $y \in \mathbb{R}$.

a.) WTS: $\int |h(x,y)| < \infty$

$$\int |h(x,y)| = \int\int |f(x)\cdot f(x+y)|\, dy\, dx = \int\int |f(x)|\cdot|f(x+y)|\,dy\,dx$$

$$= \int |f(x)| \underbrace{\left(\int |f(x+y)|\, dy\right)}_{= \|f\|_{L^1} < \infty} dx = \int C\,|f(x)|\,dx = C\int |f(x)|\,dx = C^2 < \infty \checkmark$$

$\hookrightarrow$ say $\|f\|_{L^1} = C < \infty$

as leb measure is translation invariant

b.) $x \mapsto f(x)f(x+y) = h^y(x)$

By part (a.), $h(x,y) \in L^1(\mathbb{R}^2, m^2)$ so $h^y(x) \in L^1(\mathbb{R}, m)$ for a.e. y.
Then $h^y(x)$ is a function for (leb.) a.e. y

c.) $x \mapsto g(x)g(x+y)$ for at least one y

$$g(x) = \begin{cases} \sqrt{x} & x \in (0,1] \\ 0 & \text{else} \end{cases}$$

$y = 0 \Rightarrow g(x)g(x+y) = g(x)^2 = \underbrace{\frac{1}{x} \text{ on } (0,1]}_{\text{NOT } L^1}$ ☆

FALL 2016

1. Let X denote the set of all continuous real-valued functions $f : [0,1] \to \mathbb{R}$. For $f, g \in X$, define

$$d(f,g) = \max\left\{ |f(x) - g(x)| : 0 \le x \le 1 \right\}.$$

a. **Prove** that d is a metric on X and that (X, d) is a complete metric space.

b. Let $\mathbf{0}$ denote the function in X which is identically equal to zero, and let $B = \{f \in X : d(f, \mathbf{0}) \le 1\}$. **Prove** that B is not compact. HINT: In a metric space, compactness is equivalent to sequential compactness.

$\max\{\underbrace{|\cdot|}_{\ge 0}\} \ge 0$

so $d(f,g) \ge 0$

a.) $d(f,f) = \max\{|f(x) - f(x)| \mid 0 \le x \le 1\} = \max\{0\} = 0$ ✓

$d(f,g) = \max\{|f(x) - g(x)| \mid 0 \le x \le 1\}$

$|f(x) - g(x)| = |g(x) - f(x)|$

$= \max\{|g(x) - f(x)| \mid 0 \le x \le 1\} = d(g,f)$

$d(f,h) = \max\{|f(x) - h(x)| \mid 0 \le x \le 1\}$

$|f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)|$

$\le |f(x) - g(x)| + |g(x) - h(x)|$

$\le \max\{|f(x) - g(x)| + |g(x) - h(x)| \mid 0 \le x \le 1\}$

$\le \max\{|f(x) - g(x)|\} + \max\{|g(x) - h(x)|\}$

$= d(f,g) + d(g,h)$ ✓

For complete, every Cauchy seq. converges

let $\{f_n\}$ be Cauchy seq., i.e. $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n,m \ge N$, $|f_n - f_m| < \varepsilon$

$|f_n(x) - f_m(x)| \le \underbrace{d(f_n, f_m)}_{<\varepsilon}$ $\forall x \in [0,1]$

$\Rightarrow |f_n(x) - f_m(x)| < \varepsilon$ $\forall x \in [0,1]$

$\{f_n(x)\}$ is a numerical Cauchy seq, so has a limit $f(x)$.

construct f by considering $f(x)$ at each $x \in [0,1]$

↳ Then we will get that $f_n \to f$ uniformly

Know: $|f_n(x) - f_m(x)| < \varepsilon$ $\forall m,n \ge N$

let $m \to \infty$, $|f_n(x) - f(x)| < \varepsilon$ $\forall x$

$\Rightarrow f_n \to f$ uniformly

uniform conv. $\Rightarrow f$ is cont, so $f \in X$

$|f_n(x) - f(x)| < \varepsilon$ $\forall x$

$\Rightarrow d(f_n, f) < \varepsilon$ $\forall n > N$

b.) $B = \{f \in X \mid d(f, \mathbf{0}) \le 1\}$

cpt = seq. cpt in metric space

seq cpt $\Rightarrow$ every seq has conv. subseq. in X

$\{f_n\} = x^n$ for $x \in [0,1]$

$f_n \to f = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$ But f is discont.

subseq must converge to the same thing

But limit discont, so no subseq of $\{f_n\}$ converges in X

2. Let a, b be real numbers such that $a < b$, and let $f : [a, b] \to \mathbb{R}$.

a. ***Define*** what it means for f to be "absolutely continuous on $[a, b]$" (this is an ε-δ definition).

b. State a theorem relating the absolute continuity of such a function to its differentiability. FTC

c. Assume that the restriction $f|_{[\varepsilon,1]}$ is absolutely continuous for every ε such that $0 < \varepsilon < 1$, and that $\int_0^1 x^2 |f'(x)|^p dx < \infty$ for some real number p such that $p > 3$. Prove that $\lim_{x\to 0} f(x)$ exists and is finite. (HINT: Prove that $\int_0^1 |f'(x)| dx < \infty$.)

a.) $\forall \varepsilon > 0$ + any finite collection of intervals $\{(a_i, b_i)\}_{i=1}^n$ (pg.105)

$\sum_{i=1}^n |a_i - b_i| < \delta \Rightarrow \sum_{i=1}^n |F(a_i) - F(b_i)| < \varepsilon$

b.) FTC: f is abs cont on $[a,b]$ $\Leftrightarrow$

$f(x) = f(a) + \int_a^x f'(t)\,dt$ (and f diff'able a.e. on $[a,b]$)

*Similar to Fall 2015

c.) $f(1) = f(\varepsilon) + \int_\varepsilon^1 f'(t)\,dt$ for $f|_{[\varepsilon,1]}$ abs cont

$$\int_0^1 |f'(x)|\,dx = \int_0^1 |f'(x)^p|^{1/p} |x^2|^{1/p} |x^2|^{-1/p}\,dx$$

$$= \int_0^1 |x^2 f'(x)|^{1/p} |x^2|^{-1/p}$$

$$= \| |x^2 f'(x)|^{1/p} \|_p \cdot \| |x^2|^{-1/p} \|_{p/p-1}$$

By Holder: $\frac{1}{p} + \frac{p-1}{p} = 1$

$$= \underbrace{\left(\int_0^1 |x^2 f'(x)|\right)^{1/p}}_{<\infty \text{ (given)}} \cdot \underbrace{\left(\int_0^1 |x^2|^{-\frac{1}{p}\cdot\frac{p}{p-1}}\right)^{\frac{p-1}{p}}}_{= \left(\int_0^1 |x^2|^{\frac{-1}{p-1}}\right)^{\frac{p-1}{p}}} < \infty \cdot \infty = \infty \checkmark$$

↳ $p > 3$, so $p - 1 > 2$

$\int_0^1 x^{\underbrace{-\frac{2}{p-1}}_{<1}} < \infty$ ✓

$$\lim_{\varepsilon \to 0} f(\varepsilon) = f(1) - \underbrace{\int_0^1 f'(x)\,dx}_{<\infty} \Longrightarrow \lim_{\varepsilon \to 0} f(\varepsilon) < \infty \checkmark$$

must be $<\infty$ since f is cont on $[\varepsilon, 1]$

3. Let n be a positive integer.

a. ***Define*** what it means for a subset S of $\mathbb{R}^n$ to be "connected".

b. Let Ω be an open connected subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be a function such that

$$\lim_{\varepsilon \to 0} \frac{f(p+\varepsilon v) - f(p)}{\varepsilon} = 0$$

for every $p \in \Omega$ and every $v \in \mathbb{R}^n$. ***Prove*** that f is a constant. Make sure that you use in this proof the definition of "connected" that you gave in Part a.

a.) $\nexists$ X, Y s.t. X + Y are nonempty, open, disjt. + $S = X \cup Y$

b.) To show const, WTS: for any 2 pts $x, y \in \mathbb{R}^n$, f const on straight line btwn them

$g(t) = f((1-t)x + ty)$ for $t \in [0,1]$ (param. path)

$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$

$1-t+h$

$g(t-h) - g(t) = f((1-(t-h))x + (t-h)y) - f((1-t)x + ty)$

$= f((1-t)x + hx + ty - hy) - f((1-t)x + ty)$

$= p$ $\quad = hv$

Then $g'(t) = 0$ $\forall t \in [0,1]$ by limit def. in the question

$\Rightarrow$ g is constant on [0,1]

$\Rightarrow$ f is constant on line from x to y

Gotta use connectedness...

Let $D \subset \Omega$ be a disk. f must be const on disk (consider path from center of disk c to any other pt. p, f const so $f(c) = f(p)$ $\forall p \in D$) say $f(c) = a$, so $f \equiv a$ on D.

Let $D_a = \{p \mid f(p) = a\}$. This is open, (any pt. within $B_\varepsilon(p)$ can be connected to p by line segment along radius of ball, so must also have $f(p_0) = a$, so $B_\varepsilon(p) \subset D_a$)

consider $\bigcup_{b \neq a} D_b$. Then $\bigcup D_b$ is open (union of open sets) + disjt from D_a, and $D_a \cup (\bigcup D_b) = \Omega$. But Ω is connected, so one of D_a or D_b is empty. $c \in D_a$, so $(\bigcup D_b) = \emptyset$, and thus:

$D_a \cup \emptyset = \Omega \Rightarrow D_a = \Omega$

$\Rightarrow \Omega = \{p \mid f(p) = a\}$

$\Rightarrow f \equiv a$ on Ω (const.)

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that

$$\int_{-\infty}^{\infty} (1 + |x|)|f(x)|dx < \infty .$$

Define

$$g(y) = \int_{-\infty}^{\infty} f(x) \cos(xy)dx .$$

1. ***Prove*** that g is continuously differentiable (that is, prove that the derivative $g'(y) = \lim_{h\to 0} \frac{g(y+h)-g(y)}{h}$ exists for every $y \in \mathbb{R}$, and is a continuous function of y).

2. ***Write a formula*** for g', as an integral.

$g(y) = {}_{-\infty}\int^{\infty} f(x)\cos(xy)dx$

$g'(y) = \lim_{h\to 0} \frac{g(y+h)-g(y)}{h}$ WTS: exists $\forall y$ & is cont

$g(y+h) - g(y) = \int_{-\infty}^{\infty} f(x)\cos(x(y+h))\,dx - \int_{-\infty}^{\infty} f(x)\cos(xy)dx$
$= \int_{-\infty}^{\infty} f(x)\,[\cos(x(y+h)) - \cos(xy)]\,dx$

$\frac{d}{dy}[f(x)\cos(x(y+h))] = f(x)\sin(x(y+h))\cdot x$ bdd by L^1 func! can use DCT...

Note: $f(x)\cos(xy) \le (1+|x|)\,|f(x)|$
$f(x) \le |f(x)|$
$\cos(xy) \le 1 \le (1+\underbrace{|x|}_{\ge 0})$

Then since $\int f(x)\cos(xy)dx \le \int (1+|x|)|f(x)|\,dx < \infty$ conclude that $f(x)\cos(xy) \in L^1$

Let $j_h = f(x)[\cos(x(y+h)) - \cos(xy)]$. Then $\{j_h\} \subseteq L^1$ since each $\in L^1$, & two L^1 funcs added is still L^1. By DCT:

$\lim_{h\to 0} \frac{1}{h}(g(y+h) - g(y)) = \lim_{h\to 0} \frac{1}{h}\int f(x)[\cos(x(y+h)) - \cos(xy)]\,dx$
$= \int \lim_{h\to 0} \frac{1}{h}\cdot f(x)[\cos(x(y+h)) - \cos(xy)]\,dx$
$= \int g'(y)$
$= \int x\cdot f(x)\sin(xy)dx$

By DCT, $\lim_{h\to 0}\{j_h\} = j = x\cdot f(x)\sin(xy)$ satisfies $j \in L^1$.
Then $g'(y) = \int j\,dx$ exists $(<\infty)$ everywhere (i.e. $\forall y$)

5. Let T be a real number such that $T > 0$. Let $f : (0,T) \to \mathbb{R}$ be a Lebesgue integrable function. (Here $(0,T)$ is the open interval $\{x \in \mathbb{R} : 0 < x < T\}$.) Define a function $g : (0,T) \to \mathbb{R}$ by letting

$$g(x) = \int_x^T \frac{f(t)}{t}\,dt\,.$$

Prove that g is integrable on $(0,T)$ (1) and $\int_0^T g(x)dx = \int_0^T f(x)dx$ (2).
HINTS: (a) You may want to consider first the case in which f is nonnegative.
(b) Use the Fubini-Tonelli theorem.)

For Fubini-Tonelli, need non-neg, msable on $[0,T] \times [0,T]$

WTS: $\int_0^T g(x)dx = \int_0^T f(x)\,dx$ (2)

since f Leb. intable on $[0,T]$, proving this also gives (1)

Let $h(x,t) = \begin{cases} f(t)/t & x < t \\ 0 & x \geq t \end{cases}$

use F-T on $\int_0^T \int_0^T h(x,t)\,dx\,dt$

$\int_0^T \int_0^T h(x,t)\,dt\,dx = \int_0^T \underbrace{\int_x^T f(t)/t\,dt}_{=g(x)}\,dx = \int_0^T g(x)\,dx$

$=$ by F-T ✓

$\int_0^T \int_0^T h(x,t)\,dx\,dt = \int_0^T \int_0^t f(t)/t\,dx\,dt$
$= \int_0^T \frac{f(t)}{t} \underbrace{\int_0^t dx}_{=(t-0)=t}\,dt = \int_0^T \frac{f(t)}{t}\cdot t\,dt = \int_0^T f(t)\,dt$

Note: For h to be nonneg so F-T applies, assume f nonneg.
we can write $f = f_+ - f_-$ for f_+, f_- nonneg.
Then we can apply this argument to each of f_+, f_-:

$g_+ = \int_0^T \frac{f_+(t)}{t}\,dt$

$g_- = \int_0^T \frac{f_-(t)}{t}\,dt$

$g = g_+ - g_- = \int_x^T \frac{f_+(t) - f_-(t)}{t}\,dt$

Thus, get g nonneg + intable by showing g_+ + g_- each are.

SPRING 2017

1. The sum $A+B$ of two subsets of $\mathbb{R}^n$ is $A+B=\{x+y;\ x\in A,\ y\in B\}$.

a) Show that if A is closed and B is compact, then $A+B$ is closed.

b) Show that the sum $A+B$ of two compact subsets of $\mathbb{R}^n$ is compact.

c) Show that the sum of two closed sets is not necessarily closed.

a.) A closed, B cpt $\Rightarrow A+B$ closed

A closed $\Rightarrow \forall \{x_n\}\subseteq A$ w/ $x_n\to x$, $x\in A$

B cpt $\Rightarrow \forall\{y_n\}\subseteq B$, $\exists\{y_{n_k}\}$ s.t. $y_{n_k}\to y\in B$

Let $\{z_n\}\subseteq A+B$ w/ $\{z_n\}\to z$. ($\underline{\text{WTS}}$: $z\in A+B$)

Note $\forall n$, $z_n = x_n + y_n$ for some $\{x_n\}\subseteq A$ + $\{y_n\}\subseteq B$

$\exists\{y_{n_k}\}\to y\in B$

Then $\{x_{n_k}\}\to x$ (subseq. has same lim pt as seq.)

$\Rightarrow z_{n_k} = x_{n_k}+y_{n_k}\ \ \forall n_k$

$\lim z_{n_k} = \lim(x_{n_k}+y_{n_k}) = \lim x_{n_k} + \lim y_{n_k} = \underbrace{x}_{\in A}+\underbrace{y}_{\in B} = z \in A+B$ ✓

b.) A,B cpt $\Rightarrow A+B$ cpt in $\mathbb{R}^n$

$\{x_n\}\subseteq A \Rightarrow \exists\{x_{n_k}\}\to x\in A$

$\{y_n\}\subseteq B \Rightarrow \exists\{y_{n_k}\}\to y\in B$

Consider $\{z_n\}\subseteq A+B$. Extract subseq $\{z_{n_k}\}$

↳ $z_n = x_n+y_n\ \forall n$ ↳ $z_{n_k} = x_{n_k}+y_{n_k}\ \forall n_k$

Then $\lim z_{n_k} = \lim(x_{n_k}+y_{n_k}) = \lim x_{n_k} + \lim y_{n_k} = x+y = z\in A+B$ ✓

So any seq in $A+B$ has conv subseq in $A+B$

c.) A,B closed, but $A+B$ not necessarily closed

$\boxed{\text{Counterex:}}$ **cR: AJ**

$A=\mathbb{N}$

$B=\{-n+\frac{1}{n}\mid n\in\mathbb{N}\}$

Seq: $\{z_n\}=\{\frac{1}{n}\}\subseteq A+B$

but $\lim\frac{1}{n}=0\notin A+B$

2. For any $a > 0$, show that $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges absolutely and uniformly to a continuous function on $[0, a]$. Show that

$$\int_0^x f(y)\,dy = \sum_{n=0}^{\infty} \int_0^x \frac{\sin(ny)}{n^2}\,dy$$

and that the right-hand side converges uniformly to the the left-hand side on $[0, a]$.

Weierstrass M-Test

If $\exists$ a seq. of nonneg (M_n) s.t. $|f_n(x)| \le M_n$ $\forall n \ge 1$, $x \in A$, and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ conv abs + uni on A

Then $f_n(x) = \frac{\sin(nx)}{n^2}$

$M_n = \frac{1}{n^2}$ since $\sin(nx) \in [-1,1]$, thus

$|f_n(x)| = \left|\frac{\sin(nx)}{n^2}\right| = \frac{1}{n^2} \underbrace{|\sin(nx)|}_{\in [0,1]} \le \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the Weierstrass M-Test yields convergence of $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ both abs + uni ✓ on $[0,a]$ $\forall a > 0$

Since each f_n is cont, the limit f is also cont (by Weierstrass)

Next, show that $\int_0^x f(y)\,dy = \sum_{n=0}^{\infty} \int_0^x \frac{\sin(ny)}{n^2}\,dy$

$\int_0^x f(y)\,dy = \int_0^x \sum_{n=0}^{\infty} \frac{\sin(ny)}{n^2}\,dy = \sum_{n=0}^{\infty} \int_0^x \frac{\sin(ny)}{n^2}\,dy$

swapping sum + int is a special case of Fubini-Tonelli

To use Fubini-Tonelli, need $g(y) = \frac{\sin(ny)}{n^2}$ to satisfy $g \in L^1([0,x], \mu(y))$ and $\underbrace{\sum_{n=0}^{\infty} |g(y)| < \infty}_{\text{true by abs conv ✓}}$

To show $g \in L^1([0,x])$:

$\int_0^x |g(y)|\,dy = \int_0^x \left|\frac{\sin(ny)}{n^2}\right| dy$

$= \int_0^x \frac{1}{n^2} |\sin(ny)|\,dy$

$\le \int_0^x \frac{1}{n^2}\,dy$

$= \frac{1}{n^2} \mu([0,x]) = \frac{x}{n^2} < \infty$ ✓

Then swapping sum + int is allowed, + we have satisfied the equality

Since $\sum_{n=0}^{\infty} \int_0^x \frac{\sin(ny)}{n^2}\,dy = \int_0^x \underbrace{\sum_{n=0}^{\infty} \frac{\sin(ny)}{n^2}}_{\text{conv uni to } f(y)}$, RHS $\rightrightarrows \int_0^x f(y)\,dy$ ✓ as desired

3. a) Let m denote Lebesgue measure on the bounded interval $[a, b]$. Show that if $\{f_n; n \geq 1\}$ is a sequence of real-valued, Lebesgue measurable functions on $[a, b]$ and $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere with respect to Lebesgue measure, then f_n converges to f in (Lebesgue) measure.
b) Show that in general, convergence of a sequence of functions in measure on a finite interval $[a, b]$ does not imply convergence almost everywhere.

a.) $f_n(x) \to f(x)$ a.e. $\Rightarrow$ conv in μ

Idea: finite, bdd set, so any subset of "far away" pts will have finite measure + can be made "small"

$f_n(x) \to f(x)$ a.e. on finite measure set

$\Rightarrow$ Egorov: $\forall \varepsilon > 0,\ \exists\ E \subseteq [a,b]$ s.t. $\mu(E) < \varepsilon$ + $f_n \rightrightarrows f$ on E^c

$f_n \rightrightarrows f \Rightarrow \forall \varepsilon > 0,\ \exists N$ s.t. $\forall n \geq N,\ |f_n(x) - f(x)| < \varepsilon \quad \forall x \in E^c$

Then $\{x \mid |f_n(x) - f(x)| > \varepsilon\} \subseteq E$

$\Rightarrow \mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) \leq \mu(E) < \varepsilon$ by monotonicity of measure

Hence $\forall \varepsilon > 0,\ \mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$, so $f_n \xrightarrow{\mu} f$ ✓

b.) Counterexample: Typewriter seq on $[0,1]$

$f_n(x) = \chi_{E_n}(x)$

$E_1 = [0, \frac{1}{2}],\ E_2 = [\frac{1}{2}, 1]$

$E_3 = [0, \frac{1}{3}],\ E_4 = [\frac{1}{3}, \frac{2}{3}],\ E_5 = [\frac{2}{3}, 1]$ etc.

Then $f_n \to 0$ in measure, but $f_n \not\to 0$ ptwise

as for any $x \in [0,1]$, there are ∞ many n s.t. $f_n(x) = 0$ and ∞ many s.t. $f_n(x) = 1$

Hence $\nexists N$ s.t. $\forall n \geq N,\ |f_n(x) - 0| < \varepsilon$

↳ or for any limit func f, for that matter

4. Let a, b be real numbers such that $a < b$.

(i). Define what it means for a function $f : [a, b] \mapsto \mathbb{C}$ to be "absolutely continuous".

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $\sum_{i=1}^{N} |b_i - a_i| < \delta$ (for disjoint $[a_i, b_i] \subseteq [a, b]$)
then $\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \varepsilon$

(ii). ***Prove***, using the definition of absolute continuity given in Part 1, that the product of two absolutely continuous functions is absolutely continuous.

f, g are abs cont. Let $h(x) = f(x) \cdot g(x)$.
Let $\varepsilon > 0$. Pick δ_f s.t. for any set of disjoint intervals $[a_i, b_i], i \in [1, N]$,
$\sum_{i=1}^{N} |b_i - a_i| < \delta_f \Rightarrow \sum_{i=1}^{N} |f(b_i) - f(a_i)| < \varepsilon$.

Pick δ_g s.t. the same is true for g, + let $\delta = \min\{\delta_f, \delta_g\}$. Thus whenever $\sum_{i=1}^{N} |b_i - a_i| < \delta$,
both $\sum_{i=1}^{N} |f(b_i) - f(a_i)|$, $\sum_{i=1}^{N} |g(b_i) - g(a_i)| < \varepsilon$

Then, consider:

$$\begin{aligned}
\sum_{i=1}^{N} |h(b_i) - h(a_i)| &= \sum_{i=1}^{N} |g(b_i) \cdot f(b_i) - g(a_i) \cdot f(a_i)| \\
&= \sum_{i=1}^{N} |g(b_i) \cdot f(b_i) - g(a_i) \cdot f(b_i) + g(a_i) \cdot f(b_i) - g(a_i) \cdot f(a_i)| \\
&= \sum_{i=1}^{N} |(g(b_i) - g(a_i)) \cdot f(b_i) + g(a_i)(f(b_i) - f(a_i))| \\
&\leq \sum_{i=1}^{N} |g(b_i) - g(a_i)| \cdot |f(b_i)| + |g(a_i)| \cdot |f(b_i) - f(a_i)| \\
&\leq \sum_{i=1}^{N} |g(b_i) - g(a_i)| \cdot C_1 + \sum_{i=1}^{N} C_2 \cdot |f(b_i) - f(a_i)| \\
&< \varepsilon \cdot C_1 + C_2 \cdot \varepsilon = \varepsilon(C_1 + C_2)
\end{aligned}$$

where $C_1 = \max\{f(b_i) \mid i \in [1, N]\}$
$C_2 = \max\{g(a_i) \mid i \in [1, N]\}$

Since C_1, C_2 each finite as f is continuous, we have satisfied the definition for h being abs cont.

5. In this problem, m_2 is the two-dimensional Lebesgue measure. We want to study the double integral

$$I(T) = \int_0^\infty \int_0^T e^{-xy} \sin x \, dx \, dy ,$$

that is,

$$I(T) = \int_{\mathbb{R}\times\mathbb{R}} e^{-xy} \sin x \, \chi_{E(T)}(x,y) \, dm_2(x,y) , \qquad (1)$$

where $\chi_{E(T)}$ is the characteristic function of the set

$$E(T) = \{(x,y) \in \mathbb{R}\times\mathbb{R} : 0 \leq x \leq T \text{ and } y \geq 0\} .$$

(i). **Prove** that the integrand of (1) (that is, the function $f_T : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ given by

$$f_T(x,y) = e^{-xy} \sin x \, \chi_{E(T)}(x,y)$$

for $(x,y) \in \mathbb{R}\times\mathbb{R}$) is integrable. (HINT: $|\sin x| \leq |x|$.)

(ii). By computing $I(T)$ in two different ways, ***prove*** that

$$\lim_{T\to+\infty} \int_0^T \frac{\sin x}{x} dx = \frac{\pi}{2} ;$$

Justify all the steps rigorously.

NOTE: You are allowed to use the facts that

$$\int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{2} ,$$

and

$$\int e^{-ax} \sin(x) \, dx = \frac{-e^{-ax}(\cos(x) + a\sin(x))}{1+a^2} + C .$$

i.) $I(T) = \int_{\mathbb{R}\times\mathbb{R}} e^{-xy} \sin x \, \chi_{E(T)}(x,y) \, dm_2(x,y)$

$\int_{\mathbb{R}\times\mathbb{R}} |f_T(x,y)| \, dm_2(x,y) = \int_{\mathbb{R}\times\mathbb{R}} |e^{-xy} \sin x \, \chi_{E(T)}| \, dm_2$

$= \int_{\mathbb{R}\times\mathbb{R}} |e^{-xy}| \cdot |\sin x| \cdot |\chi_{E(T)}| \, dm_2$

$E(T) = [0,T] \times [0,\infty) \Rightarrow \chi_{E(T)} \neq 0$ only for $x,y \geq 0$

$\Rightarrow e^{-xy} = \frac{1}{\underbrace{e^{xy}}_{\geq 0}} \leq 1$ since $e^z \geq 1 \ \forall z \geq 0$

So: $|e^{-xy}| \leq 1$ when $\chi_{E(T)} \neq 0$, $|\sin x| \leq |x|$

$\int_{\mathbb{R}\times\mathbb{R}} |f_T(x,y)| \, dm_2 \leq \int_{E(T)} 1 \cdot |x| \cdot 1 \, dm_2 = \int_{E(T)} \underbrace{x}_{\hookrightarrow x \geq 0 \text{ on } E(T)} \, dm_2$

$= (T - 0) = T < \infty$ ✓

?

so $f_T(x,y)$ is integrable ✓

ii.) Since $f(x,y) \in L^+$ (nonneg due to def of $E(T)$, integrable by (i.)), Fubini-Tonelli applies. So we compute $\int_{\mathbb{R}\times\mathbb{R}} f_T(x,y) \, dm_2$ in two (equivalent) ways:

Note: Since f is nonneg, $g_T = \int_0^T e^{-xy} \sin x \, dx$ is nondec, so monotone.

$\lim_{T\to\infty} \int_{\mathbb{R}} e^{-xy} \sin \chi_{E(T)} \, dx = \int_{\mathbb{R}} \lim_{T\to\infty} e^{-xy} \sin x \, \chi_{E(T)} \, dx = \int_0^\infty e^{-xy} \sin x \, dx$

• INTEGRATE WRT X FIRST

$\lim_{T\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-xy} \sin x \, \chi_{E(T)} \, dx \, dy = \int_0^\infty \int_0^\infty e^{-xy} \sin x \, dx \, dy = \int_0^\infty \frac{1}{1+y^2} dy = \frac{\pi}{2}$

$\int_0^\infty e^{-xy} \sin x \, dx = \frac{-e^{-xy}(\cos x + y \sin x)}{1+y^2} \Big|_{x=0}^{x=\infty}$

$= \frac{1}{1+y^2} \left[(-\underbrace{e^{-\infty}}_{0} (\underbrace{\cos(\infty)}_{\leq 1} + y \underbrace{\sin(\infty)}_{\leq 1})) - (\underbrace{-e^0}_{-1} (\underbrace{\cos(0)}_{1} + y \underbrace{\sin(0)}_{0})) \right] = \frac{1}{1+y^2}(0 - -1(1+0)) = \frac{1}{1+y^2}$

• INTEGRATE WRT Y FIRST

$\lim_{T\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-xy} \sin x \, \chi_{E(T)} \, dy \, dx = \lim_{T\to\infty} \int_0^T \int_0^\infty e^{-xy} \sin x \, dy \, dx = \lim_{T\to\infty} \int_0^T \sin x \int_0^\infty e^{-xy} \, dy \, dx$

$= \lim_{T\to\infty} \int_0^T \sin x \, \underbrace{\left[\frac{-1}{x} e^{-xy}\right]_{y=0}^{\infty}} \, dx = \lim_{T\to\infty} \int_0^T \sin x \cdot \frac{1}{x} \, dx$

$= \frac{-1}{x}(\underbrace{e^{-\infty}}_{0} - \underbrace{e^{-0}}_{1}) = \frac{1}{x}$

Setting these 2 integrals equal to each other (using Fubini-Tonelli):

$$\lim_{T\to\infty} \int_0^T \frac{\sin x}{x} dx = \frac{\pi}{2} \quad \text{as desired!}$$

FALL 2017

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^1(X, \mathcal{M}, \mu)$ be such that $f(x) > 0$ almost everywhere. Let $E \in \mathcal{M}$ be such that

$$\int_E f \, d\mu < \infty .$$

Prove that

$$\lim_{k \to +\infty} \int_E f^{1/k} d\mu(x) = \mu(E).$$

would love to pull limit inside.

Define seq. $\{g_k\} = \{f^{1/k}\}$, then $\lim g_k = \lim f^{1/k} = f^0 = 1$ a.e.

↳ since $f(x) > 0$ and $f \in L^1$, note $\{g_k\} \subseteq L^+$

let $E_1 = \{x \in E \mid f(x) \leq 1\}$ then $E = E_1 \sqcup E_2$, so $\mu(E_1) + \mu(E_2) = \mu(E)$

$E_2 = \{x \in E \mid f(x) > 1\}$ ↳ disjt

On E_1, $g_k(x) = f(x)^{1/k} \leq f(x)^{1/k+1} = g_{k+1}(x)$ ⇒ can use MCT!

By MCT:

$\lim \int_{E_1} f^{1/k} d\mu = \lim \int_{E_1} g_k \, d\mu \underset{(MCT)}{=} \int_{E_1} \lim g_k \, d\mu = \int_{E_1} 1 \, d\mu = \mu(E_1)$

On E_2, $|g_k(x)| = |f(x)^{1/k}| \underset{(f(x) > 0)}{=} f(x)^{1/k} < f(x) \in L^1$ ⇒ can use DCT!

$\Rightarrow \lim \int_{E_2} f^{1/k} d\mu = \lim \int_{E_2} g_k \, d\mu \underset{(DCT)}{=} \int_{E_2} \lim g_k \, d\mu = \int_{E_2} 1 \, d\mu = \mu(E_2)$

Then all together:

$\lim \int_E f^{1/k} d\mu = \lim \left(\int_{E_1} f^{1/k} d\mu + \int_{E_2} f^{1/k} d\mu \right) = \mu(E_1) + \mu(E_2) = \mu(E)$ ✓

2. Let X be a set, and let μ^* be an outer measure on X such that $\mu^*(X) < \infty$. Define ν^* by letting

$$\nu^*(E) = \sqrt{\mu^*(E)} \qquad \text{for every} \quad E \subseteq X.$$

Prove that

1. ν^* is an outer measure.
2. A subset $A \subseteq X$ belongs to the Carathéodory σ-algebra of ν^* (the σ-algebra of ν^*-measurable sets) if and only if either $\nu^*(A) = 0$ or $\nu^*(A^c) = 0$.

1.) For an outer measure, we need:

i.) $\nu^*(\emptyset) = 0$

$\nu^*(\emptyset) = \sqrt{\mu^*(\emptyset)} = \sqrt{0} = 0$ ✓

ii.) monotonicity

Let $B \subseteq A$. Then:

$\nu^*(B) = \sqrt{\mu^*(B)} \leq \sqrt{\mu^*(A)} = \nu^*(A)$ ✓

iii.) subadditivity

$\nu^*(\bigcup_{i=1}^{\infty} E_i) = \sqrt{\mu^*(\bigcup_{i=1}^{\infty} E_i)} \leq \sqrt{\sum_{i=1}^{\infty} \mu^*(E_i)} \leq \sum_{i=1}^{\infty} \sqrt{\mu^*(E_i)} = \sum_{i=1}^{\infty} \nu^*(E_i)$ ✓

2.) ($\Rightarrow$) Let A be ν^*-measurable. Then $\forall E \subset X$,

$\nu^*(E) = \nu^*(E \cap A) + \nu^*(E \cap A^c)$

LHS: $\nu^*(E) = \sqrt{\mu^*(E)} = \sqrt{\mu^*(E \cap A) + \mu^*(E \cap A^c)}$

RHS: $\nu^*(E \cap A) + \nu^*(E \cap A^c) = \sqrt{\mu^*(E \cap A)} + \sqrt{\mu^*(E \cap A^c)}$

$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

$\Rightarrow a + b = a + b + 2\sqrt{ab}$

$\Rightarrow 0 = 2\sqrt{ab} \Rightarrow$ at least one of $a, b = 0$

Since LHS = RHS, at least one of $\mu^*(E \cap A), \mu^*(E \cap A^c) = 0$ (see note above)

This is true $\forall E \subset X$, including $E = A$ or $E = A^c$

If $\mu^*(E \cap A) = 0$, then $\mu^*(A \cap A) = \mu^*(A) = 0$

$\Rightarrow \nu^*(A) = \sqrt{\mu^*(A)} = \sqrt{0} = 0$ ✓

If $\mu^*(E \cap A) = 0$, then $\mu^*(A^c \cap A^c) = \mu^*(A^c) = 0$

$\Rightarrow \nu^*(A^c) = \sqrt{\mu^*(A^c)} = \sqrt{0} = 0$ ✓

Then either $\nu^*(A) = 0$ or $\nu^*(A^c) = 0$ ✓

($\Leftarrow$) Assume that either $\nu^*(A) = 0$ or $\nu^*(A^c) = 0$.

WLOG let $\nu^*(A) = 0$. Then $\forall E \subset X$, $\nu^*(E \cap A) = 0$ by monotonicity of outer measure.

Note that $E = (E \cap A) \cup (E \cap A^c)$, so subadditivity gives us that

$\nu^*(E) \leq \nu^*(E \cap A) + \nu^*(E \cap A^c) = \nu^*(E \cap A^c)$

Additionally, since $(E \cap A^c) \subset E$, monotonicity of ν^* again gives us that $\nu^*(E \cap A^c) \leq \nu^*(E)$.

With both ineq., we have that $\nu^*(E) = \nu^*(E \cap A^c)$. Then for any $E \subset X$, we get:

$\nu^*(E) = \nu^*(E \cap A^c) + 0$

$\nu^*(E) = \nu^*(E \cap A^c) + \nu^*(E \cap A)$

$\Rightarrow A$ is ν^*-measurable! ✓

3. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function. Define a function $f : \mathbb{R} \mapsto \mathbb{C}$ by letting

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{ixy}}{1+y^2+\sigma(y)}\,dy. \tag{1}$$

Prove that

a.) 1. The function f is well defined (that is, the integral of (1) exists) and continuous for all $x \in \mathbb{R}$. $< \infty$

or can mimic Thm 2.27 pf in Folland (use DCT)

b.) 2. If we make the additional assumption that σ is bounded or integrable, then f is differentiable on $\mathbb{R}\setminus\{0\}$ but is not differentiable at 0. You are allowed to use the identity

$$\int_{-\infty}^{\infty} \frac{e^{ixy}}{1+y^2}\,dy = \pi e^{-|x|},$$

valid for all real x.

similar to Thm. 2.27b

a.) $\int_{-\infty}^{\infty} \left|\frac{e^{ixy}}{1+y^2+\sigma(y)}\right| dy = \int_{-\infty}^{\infty} \frac{1}{|1+y^2+\sigma(y)|}\,dy \le \int_{-\infty}^{\infty} \frac{1}{|1+y^2|}\,dy = \int_{-\infty}^{\infty} \underbrace{\frac{1}{1+y^2}}_{\in L^1}\,dy < \infty$ ✓ $(=\pi)$

$\sigma(y) \ge 0$, so $|1+y^2+\sigma(y)| \ge |1+y^2| = 1+y^2$ ($y^2 \ge 0$)

For f cont, need $\forall \varepsilon>0, \exists \delta>0$ s.t. $|x_1-x_2|<\delta$, then $|f(x_1)-f(x_2)|<\varepsilon$

Let $\varepsilon>0$, then

$$|f(x_1)-f(x_2)| = \left|\int_{-\infty}^{\infty} \frac{e^{ix_1y}}{1+y^2+\sigma(y)}\,dy - \int_{-\infty}^{\infty}\frac{e^{ix_2y}}{1+y^2+\sigma(y)}\,dy\right| = \left|\int_{-\infty}^{\infty} \frac{e^{ix_1y}-e^{ix_2y}}{1+y^2+\sigma(y)}\,dy\right|$$

Note e^{ixy} is continuous in x, so $\forall \varepsilon'>0, \exists \delta'>0$ s.t. $|x_1-x_2|<\delta' \Rightarrow |e^{ix_1y}-e^{ix_2y}|<\varepsilon'$

Let $\varepsilon' = \varepsilon/\pi$, pick corresponding δ'. Assume $|x_1-x_2|<\delta'$, then

$$|f(x_1)-f(x_2)| = \left|\int_{-\infty}^{\infty} \frac{e^{ix_1y}-e^{ix_2y}}{1+y^2+\sigma(y)}\,dy\right| \le \int_{-\infty}^{\infty} \frac{|e^{ix_1y}-e^{ix_2y}|}{|1+y^2+\sigma(y)|}\,dy < \int_{-\infty}^{\infty} \varepsilon' \cdot \frac{1}{|1+y^2+\sigma(y)|}\,dy$$

$$< \underbrace{\varepsilon' \cdot \pi}_{\text{by argument above}} = \frac{\varepsilon}{\pi}\cdot\pi = \varepsilon \checkmark$$

so $|f(x_1)-f(x_2)|<\varepsilon$ whenever $|x_1-x_2|<\delta'$, so f is continuous

b.) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ or $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{1}{x-x_0}\int_{-\infty}^{\infty} \frac{e^{ixy}-e^{ix_0y}}{1+y^2+\sigma(y)}$$

want to pull limit inside of integral. Define $g_n(x) = f(x_n)-f(x_0)$ for a seq. of pts $\{x_n\} \to x$. Then $|g_n(x)| = \left|\frac{e^{ix_ny}-e^{ix_0y}}{1+y^2+\sigma(y)}\right| = \frac{1}{|1+y^2+\sigma(y)|} \le \frac{1}{|1+y^2|} = \frac{1}{1+y^2} \in L^1$ by above argument

Then use DCT to pull in limit:

$$f'(x_0) = \lim_{x\to x_0} \frac{1}{x-x_0}\int_{-\infty}^{\infty} \frac{e^{ixy}-e^{ix_0}}{1+y^2+\sigma(y)}\,dy$$

$$= \int_{-\infty}^{\infty} \underbrace{\lim_{x\to x_0} \frac{e^{ixy}-e^{ix_0y}}{x-x_0}}_{=\frac{d}{dx}(e^{ixy}) = iy\cdot e^{ixy}} \cdot \frac{1}{1+y^2+\sigma(y)}$$

$$= i\int_{-\infty}^{\infty} \underbrace{\frac{ye^{ixy}}{1+y^2+\sigma(y)}}_{\in L^1}\,dy < \infty$$

Then the derivative exists! But if $x=0$,

$$f'(x_0) = \int_{-\infty}^{\infty} \frac{ye^0}{1+y^2+\sigma(y)}\,dy = \int_{-\infty}^{\infty} \underbrace{\frac{y}{1+y^2+\sigma(y)}}$$

doesn't converge(?) so derivative DNE @ $x=0$

4. Let (X, d) be a metric space. Prove that if (X, d) is not compact, then there exists an unbounded continuous function $f : X \to \mathbb{R}$.

(*) In metric spaces, cpt = seq cpt

↳ NOT cpt $\Rightarrow \exists$ open cover w/ no finite subcover

NOT seq cpt $\Rightarrow \exists$ seq in X w/ no conv subseq.

i.e. a seq. w/ no lim pt (else subseq must have same lim pt.)

Let $\{x_n\} \subseteq X$ where $x_n \not\to x \ \forall x \in X$

↳ use this seq to define our unbdd func.

Let $f(x_n) = n$. Clearly f is unbdd. To extend f continously to all of X, build a seq of open balls $\{B(r_n, x_n)\}$ w/ radius r_n centered @ x_n s.t. $X \subset \bigcup_{n=1}^{\infty} B(r_n, x_n)$. Then for $x \in X$, let $f(x) = \inf\{ n + \underbrace{\frac{d(x_n, x)}{r_n}}_{\text{scaled proportion of radius of the ball}} \mid B(r_n, x_n) \ni x \}$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^1(X, \mathcal{M}, \mu)$, and let $f \in L^1(X, \mathcal{M}, \mu)$. Suppose that

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0 .$$

conv. in L^1

Suppose also that

$$C := \sup\left\{ \int_X |f_n|^4 \, d\mu : n \in \mathbb{N} \right\} < \infty .$$

Prove that

1. $\int_X |f|^4 \, d\mu \le C$.
2. $\int_X |f_n|^2 \, d\mu < \infty$ for all $n \in \mathbb{N}$, and $\int_X |f|^2 \, d\mu < \infty$.
3. $\lim_{n\to\infty} \int_X |f_n - f|^2 \, d\mu = 0$.

i.) $f_n \to f$ in L^1 $\Rightarrow$ conv in μ

$\Rightarrow$ subseq conv a.e.

$\{f_{n_j}\} \to f$ a.e.

Let $g_j = |f_{n_j}|^4$, so $\{g_j\} \subseteq L^+$ & $g_j = |f_{n_j}|^4 \to |f|^4$ a.e.

By Fatou: $\int \underbrace{\liminf g_j}_{=|f|^4} \le \liminf \int g_j$

$\liminf \int g_j = \liminf \int |f_{n_j}|^4$

$\le \limsup \int |f_{n_j}|^4$

$\le \sup_n \int |f_n|^4 \overset{*}{=} C$

$\Rightarrow \int |f|^4 \le C$ ✓

ii.) $\int |f_n|^2 = \int ||f_n|^2| = \| |f_n|^2 \|_{L^1} = \| |f_n|^2 \cdot 1 \|_{L^1}$

By Holder: $\le \| |f_n|^2 \|_{L^2} \cdot \| 1 \|_{L^2}$

$= \left(\int (|f_n|^2)^2 \right)^{1/2} \cdot \left(\int 1^2 \right)^{1/2}$

$= \Big(\underbrace{\int |f_n|^4}_{< C} \Big)^{1/2} \cdot (\underbrace{\mu(X)}_{<\infty})^{1/2} \quad < \underbrace{C}_{<\infty}{}^{1/2} \cdot (\underbrace{\mu(X)}_{<\infty})^{1/2} \quad < \infty$ ✓

Same for $\int |f|^2$ since $\int |f|^4 \le C$ by part (i.)

iii.) Sneaky Holder application, cr: Leonidas

$\int |f_n - f|^2 = \| |f_n - f|^2 \|_{L^1} \le \| |f_n - f|^\alpha \|_{L^p} \cdot \| |f_n - f|^\beta \|_{L^q}$ where $\alpha + \beta = 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta > 0$

$= \left(\int |f_n - f|^{\alpha \cdot p} \right)^{1/p} \left(\int |f_n - f|^{\beta \cdot q} \right)^{1/q}$

choose α, β, p, q s.t. $\alpha \cdot p = 4$, $\beta \cdot q = 1$ $\Rightarrow \beta = 1/q$, $\alpha = 4/p$ or $\frac{\alpha}{4} = \frac{1}{p}$, so $\frac{1}{p} + \frac{1}{q} = \frac{\alpha}{4} + \beta = 1$

$\alpha + \beta = 2 \Rightarrow \beta = 2 - \alpha \Rightarrow 1 = \frac{\alpha}{4} + (2 - \alpha)$

$-1 = (\frac{1}{4} - 1)\alpha$

$\alpha = \frac{4}{3}$

$\frac{4}{3} + \beta = 2 \Rightarrow \beta = \frac{2}{3}$

$\beta = \frac{1}{q} \Rightarrow \frac{2}{3} = \frac{1}{q}$, $\frac{1}{3/2} = \frac{1}{q}$, $3/2 = q$

$\frac{\alpha}{4} = \frac{1}{p} \Rightarrow \frac{4/3}{4} = \frac{1}{p}$, $\frac{1}{3} = \frac{1}{p}$, $3 = p$

$= \left(\int |f_n - f|^4 \right)^{1/3} \cdot \left(\int |f_n - f|^1 \right)^{1/(3/2)}$

$= \left(\int |f_n - f|^4 \right)^{1/3} \cdot \left(\int |f_n - f| \right)^{2/3}$

$\le \underbrace{\| f_n - f \|_{L^4}^{4/3}}_{\le (\|f_n\|_{L^4} + \|f\|_{L^4})^{4/3} \text{ (norm)}} \cdot \| f_n - f \|_{L^1}^{2/3} \le (\underbrace{\|f_n\|_{L^4}}_{\le C^{1/4}} + \underbrace{\|f\|_{L^4}}_{\le C^{1/4}})^{4/3} \cdot \|f_n - f\|_{L^1}^{2/3} \le \underbrace{(2C^{1/4})^{4/3}}_{<\infty \text{ const}} \underbrace{\|f_n - f\|_{L^1}^{2/3}}_{\to 0} \to 0$ as $n \to \infty$

so $\int |f_n - f|^2 \to 0$, as desired

SPRING 2018

1. Let S be a connected metric space, with metric $d : S \times S \to \mathbb{R}_+$. Let $q \in S$. Show that if $S \setminus \{q\}$ is non-empty, then it is not compact.

S connected $=$ $\nexists$ disjt, open, nonempty P, Q s.t. $P \cup Q = S$
$\Rightarrow$ $S \setminus \{q\}$ and $\{q\}$ cannot both be open

SOLUTION #1

AFSOC $S \setminus \{q\}$ is both nonempty + cpt.
compact sets are closed (by Folland Prop. 0.24) so $S \setminus \{q\}$ is closed.
Then $\{q\}$ is open as the complement of a closed set.
additionally, singletons are closed in metric (Hausdorff) spaces, so $\{q\}$ is closed. Then $S \setminus \{q\}$ is open as the complement of a closed set.
Then both $S \setminus \{q\}$ and $\{q\}$ are open, which contradicts the connectedness of S. ↯
↳ Thus $S \setminus \{q\}$ cannot be both cpt + nonempty.

SOLUTION #2

CONSTRUCTION CR: RILEY

AFSOC $S \setminus \{q\}$ is both cpt. + nonempty.
Let $D = \sup\{d(q,s) \mid s \in S\}$, and define a sequence of closed balls $\{B_n(\frac{1}{n} \cdot D, q)\}$ centered at q and w/ radius $\frac{D}{n}$. Then observe that $\{B_n^c\}$ is an open cover of $S \setminus \{q\}$.
Since $S \setminus \{q\}$ is compact, there is a finite subcover $\{B_{n_k}^c \mid k \in [1,N]\}$ of this open cover. However, the B_n^c's are nested, so $B_{n_k}^c \subset B_{n_j}^c$ $\forall j \geq k$. Then $B_{n_N}^c$ must cover $S \setminus \{q\}$, as does B_m^c $\forall m \geq n_N$. Then $B_m = \{q\}$ $\forall m \geq n_N$.
By construction, our $\{B_n\}$'s were closed, so $\{q\}$ is closed and $S \setminus \{q\}$ is open as the complement of a closed set. Similarly, $S \setminus \{q\}$ is closed since it is compact, making $\{q\}$ open as its complement.
Then $S \setminus \{q\}$ and $\{q\}$ are both open and disjoint, and if $S \setminus \{q\}$ is nonempty then we contradict the connectedness of S.
Thus $S \setminus \{q\}$ cannot be both compact + nonempty!

2. Let m be the Lebesgue measure. For any $A \subseteq \mathbb{R}$, define

$$m^*(A) = \inf\{\sum_{i=1}^{\infty} m(I_i);\ \{I_i\} \text{ is a cover of } A \text{ by open intervals}\}.$$

(A) Show that m^* is an outer measure on the subsets of $\mathbb{R}$.
(B) Let A and B be subsets of $\mathbb{R}$ and assume that

$$d(A, B) := \inf\{|x - y| : x \in A, y \in B\} > 0.$$

Show that $m^*(A \cup B) = m^*(A) + m^*(B)$.

A.) For an outer measure:

i.) $m^*(\emptyset) = 0$

$$m^*(\emptyset) = \inf\{\Sigma_{i=1}^{\infty} m(I_i) \mid \cup_{i=1}^{\infty} I_i \supset \emptyset\}$$
$$= m(\emptyset)$$
$$= 0 \checkmark$$

ii.) monotonicity

let $B \subseteq A$, then

$$m^*(A) = \inf\{\Sigma_{i=1}^{\infty} m(I_i) \mid \cup_{i=1}^{\infty} I_i \supset A\}$$
$$\geq \inf\{\Sigma_{i=1}^{\infty} m(J_i) \mid \cup_{i=1}^{\infty} J_i \supset B\}$$
$$= m^*(B) \checkmark$$

If $\{I_i\}$ covers A and $B \subset A$, then $\{I_i\}$ also covers B. But there could be a cover $\{J_i\}$ of B which doesn't cover A and which is smaller in measure than all $\{I_i\}$

iii.) subadditivity

$$m^*(\cup_{j=1}^{\infty} E_j) = \inf\{\Sigma_{i=1}^{\infty} m(I_i) \mid \cup_{i=1}^{\infty} I_i \supset \cup_{j=1}^{\infty} E_j\}$$
$$\leq \Sigma_{j=1}^{\infty} \inf\{\Sigma_{i=1}^{\infty} m(I_i) \mid \cup_{i=1}^{\infty} I_i \supset E_j\}$$
$$= \Sigma_{j=1}^{\infty} m^*(E_j) \checkmark$$

B.) See Fall 24 Prob #4

3. Let $S=[0,1]\times[0,1]=[0,1]^2$, let λ denote the Lebesgue measure on the Lebesgue σ algebra $\Lambda([0,1])$, and let λ^2 denote the product measure on the product σ algebra $\Lambda\otimes\Lambda$. Let

$$f(x,y)=\frac{\cos(10x+17y)}{1-xy}.$$

Determine, with proof, whether the following integrals $I(f)$, $J(f)$ and $K(f)$ are finite, and if so, whether there holds equality between them:

$$I(f)=\int_{[0,1]}\Big(\int_{[0,1]}f(x,y)\lambda(dy)\Big)\lambda(dx),\quad J(f)=\int_{[0,1]}\Big(\int_{[0,1]}f(x,y)\lambda(dx)\Big)\lambda(dy),$$

$$K(f)=\int_S f(x,y)d\lambda^2.$$

To get full credit, you must explicitly formulate the theorem(s) you are using.

Tonelli: $f\in L^+(X\times Y)\Rightarrow f_x(y)\in L^+(Y)$
$f^y(x)\in L^+(X)$

So $\iint f\,d(\mu\times\nu)=\iint f\,d\mu(x)\,d\nu(y)$
$=\iint f\,d\nu(y)\,d\mu(x)$

Fubini: $f\in L^1(\mu\times\nu)\Rightarrow f_x(y)\in L^1(Y)$
$f^y(x)\in L^1(X)$
and Tonelli's Thm holds

$f(x,y)=\frac{\cos(10x+17y)}{1-xy}$

If $f(x,y)\in L^+([0,1]\times[0,1])$ or $L^1([0,1]\times[0,1])$, then Fubini-Tonelli says that $I(f)=J(f)$ and then also $I(f)=J(f)=K(f)$

Of course $\int_S f(x,y)\,d\lambda^2\le\int_S|f(x,y)|\,d\lambda^2$, so if $f(x,y)\in L^1([0,1]\times[0,1])$, then $K(f)$ is finite (so $I(f), J(f)$ are, too)

$\int_S|f(x,y)|\,d\lambda^2=\int_S\left|\frac{\cos(10x+17y)}{1-xy}\right|d\lambda^2=\int_S\underbrace{|\cos(10x+17y)|}_{\le 1}\cdot\frac{1}{|1-xy|}\,d\lambda^2\le\int_S\frac{1}{|1-xy|}\,d\lambda^2=\int_S\frac{1}{1-xy}\,d\lambda^2$

$x,y\in[0,1]$ so $xy\in[0,1]$
and $1-xy\in[0,1]$
$\Rightarrow|1-xy|=1-xy$

Then $|f(x,y)|\le\frac{1}{1-xy}$. So if $\int\frac{1}{1-xy}<\infty$, $\int|f|<\infty$, too!

Test: $\int_0^1\int_0^1\frac{1}{1-xy}\,dx\,dy=\int_0^1-\frac{1}{y}\ln(1-xy)\Big|_{x=0}^{x=1}dy=\int_0^1-\frac{1}{y}(\ln(1-y)-\underbrace{\ln(1)}_{=0})\,dx=\int_0^1-\frac{1}{y}\ln(1-y)\,dy$

CR: OMAR

Expand in Taylor series: $\ln(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\dots$
$\ln(1-y)=-y-\frac{(-y)^2}{2}+\frac{(-y)^3}{3}-\dots$
$=-y-\frac{y^2}{2}-\frac{y^3}{3}-\dots$
$-\ln(1-y)=y+\frac{y^2}{2}+\frac{y^3}{3}+\dots$
$=\sum_{n=1}^{\infty}\frac{y^n}{n}$

$\int_0^1\int_0^1\frac{1}{1-xy}\,dx\,dy=\int_0^1-\frac{1}{y}\ln(1-y)\,dy=\int_0^1\frac{1}{y}\sum_{n=1}^{\infty}\frac{y^n}{n}\,dy=\int_0^1\sum_{n=0}^{\infty}\frac{y^{n-1}}{n}\,dy$

$=\int_0^1 1+\frac{y}{2}+\frac{y^2}{3}+\dots\,dy=\left[y+\frac{y^2}{4}+\frac{y^3}{9}+\dots\right]\Big|_{y=0}^{y=1}=\left(1+\frac{1}{4}+\frac{1}{9}+\dots\right)-0=\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\text{converges!}}<\infty$

Woohoo! So $\frac{1}{1-xy}\in L^1([0,1]\times[0,1])$ as $\frac{1}{1-xy}=\left|\frac{1}{1-xy}\right|$.
Then since $|f(x,y)|\le\frac{1}{1-xy}$, $f(x,y)\in L^1([0,1]\times[0,1])$, too, and by Fubini, use Tonelli: $I(f)=J(f)=K(f)<\infty$ ✓

4. Let λ denote Lebesgue measure on $\mathbb{R}$. Let $x \in \mathbb{R}$, and let $f : x \mapsto f(x) \in \mathbb{R}$ be Lebesgue measurable. For Borel sets $B \subset \mathbb{R}$, define

$$\mu(B) = \lambda(\{x : f(x) \in B\}).$$

Show that μ is a measure, and that

$$\int_{\mathbb{R}} g(y)d\mu(y) = \int_{\mathbb{R}} (g \circ f)(x)d\lambda(x)$$

for all g such that the integrals make sense.

a measure has:
i.) $\mu(\emptyset) = 0$
ii.) $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$
for $\{E_i\}$ disjt sets

i.) $\mu(\emptyset) = \lambda(\{x \mid \underbrace{f(x) \in \emptyset}_{\text{no such } x,\ f(x) \notin \emptyset \text{ by def of } \emptyset}\}) = \lambda(\emptyset) = 0$ ✓

ii.) let $\{E_i\}$ a disjoint seq of sets

these sets each disjt. since f a func & E_i's disjt, $f(x)$ cannot be in E_i & E_j both

$\mu(\cup_{i=1}^{\infty} E_i) = \lambda(\{x \mid f(x) \in \cup_{i=1}^{\infty} E_i\}) = \lambda(\cup_{i=1}^{\infty} \{x \mid f(x) \in E_i\}) = \sum_{i=1}^{\infty} \lambda(\{x \mid f(x) \in E_i\}) = \sum_{i=1}^{\infty} \mu(E_i)$ ✓

$\{x \mid f(x) \in \cup_{i=1}^{\infty} E_i\} = \cup_{i=1}^{\infty} \{x \mid f(x) \in E_i\}$ since E_i's disjt

since λ is a measure

WTS: $\int_{\mathbb{R}} g(y)\, d\mu(y) = \int_{\mathbb{R}} (g \circ f)(x)\, d\lambda(x)$ Hint: approx by simple funcs

Suppose $g \geq 0$ is msable, then $\exists \{\phi_n\} \geq 0$ simple s.t. $\{\phi_n\} \nearrow g$ $\forall x \in \mathbb{R}$.

↳ If g not nonneg. split up $g = g^+ - g^-$ and apply the following argument to each g^+, g^-

↳ Note since $\phi_n \nearrow g$, w/ $\phi_n \geq 0$, $\{\phi_n\}$ is monotone

So: $\int_{\mathbb{R}} g\, d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} \phi_n\, d\mu \overset{MCT}{=} \lim_{n\to\infty} \int_{\mathbb{R}} \phi_n\, d\mu$

$x \in f^{-1}(E) \iff f(x) \in E$

Note that for $E \subseteq \mathcal{B}_{\mathbb{R}}$, $\int_{\mathbb{R}} \chi_E\, d\mu = \mu(E) = \lambda(f^{-1}(E)) = \int_{\mathbb{R}} \chi_{f^{-1}(E)}\, d\lambda = \int_{\mathbb{R}} \chi_E \circ f\, d\lambda$

So for ϕ simple, $\int_{\mathbb{R}} \phi\, d\mu = \int_{\mathbb{R}} \phi \circ f\, d\lambda$

Then $\lim_{n\to\infty} \int_{\mathbb{R}} \phi_n\, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} \phi_n \circ f\, d\lambda \overset{MCT}{=} \int_{\mathbb{R}} \lim_{n\to\infty} \phi_n \circ f\, d\lambda = \int_{\mathbb{R}} g \circ f\, d\lambda$

Hence $\int_{\mathbb{R}} g\, d\mu = \int_{\mathbb{R}} g \circ f\, d\lambda$ as desired!

5. Let $f : [0,\infty) \to \mathbb{R}$ be a Borel measurable function such that $|f(x)| \le 1 + x^2$ for all $x \ge 0$ and such that f is continuous at 0. Prove that

$$\lim_{n\to\infty} \int_0^n \frac{f(x/n)}{(1+x)^4}\,dx$$

$<\infty$

exists and calculate the limit.

CR: RILEY

would like to pull limit inside since $\lim f(x/n) = f(0) < \infty$ (since f is continuous @ 0)

$g_n(x) = f(x/n)\cdot(1+x)^{-4} \Rightarrow |g_n(x)| = \left|\frac{f(x/n)}{(1+x)^4}\right| = \frac{|f(x/n)|}{(1+x)^4} \le \frac{1+(\frac{x}{n})^2}{(1+x)^4}$

$\hookrightarrow x \in [0,\infty)$

$\int_0^\infty |g_n(x)|\,dx \le \int_0^\infty \frac{1+\frac{x^2}{n^2}}{(1+x)^4}\,dx = \int_0^\infty \frac{1}{n^2}\frac{n^2+x^2}{(1+x)^4}\,dx = \int_1^\infty \frac{1}{n^2}\frac{n+(u-1)^2}{u^4}\,du$

$u = 1+x$

$du = dx$

$= \frac{1}{n^2}\int_1^\infty \frac{n}{u^4} + \frac{u^2}{u^4} - \frac{2u}{u^4} + \frac{1}{u^4}\,du = \frac{1}{n^2}\int_1^\infty \frac{n^2+1}{u^4} - \frac{1}{u^2} - \frac{2}{u^3}\,du = \frac{1}{n^2}\left(\frac{-(n^2+1)}{3u^3} + \frac{1}{u} + \frac{1}{u^2}\right)\Big|_1^\infty$

$= \frac{1}{n^2}\left[0 - \left(\frac{-(n^2+1)}{3} + 1 + 1\right)\right] = \frac{1}{n^2}\cdot\frac{n^2+1}{3} < \infty$ ✓ $\Rightarrow \{g_n\} \subseteq L^1([0,\infty))$

let $h_n(x) = \frac{1+(x/n)^2}{(1+x)^4}$, and observe that $h_{n+1}(x) \ge h_n(x)$ a.e. $\forall n$ so $\{h_1 - h_n\}$ is a monotone seq. additionally, note that $h_n(x) \in L^+([0,\infty))$ as the product of two measurable funcs., so $\{h_1 - h_n\} \subseteq L^+([0,\infty))$. then we may use MCT:

$\lim \int h_1 - h_n$

$= \int \lim h_1 - h_n$

$= \int \lim\left(\frac{1+x^2}{(1+x)^4} - \frac{1+(\frac{x}{n})^2}{(1+x)^4}\right)$

$= \int \frac{1+x^2}{(1+x)^4} - \frac{1}{(1+x)^4}$

$= \int h_1 - \int \frac{1}{(1+x)^4}$

$\int h_1 - \int \frac{1}{(1+x)^4} = \lim \int h_1 - h_n$

$= \lim \int h_1 - \lim \int h_n$

$= \int h_1 - \lim \int h_n$

$\Rightarrow \int \frac{1}{(1+x)^4} = \lim \int h_n$

$//$

$\int \lim h_n = \int h$

$\int_0^\infty \frac{1}{(1+x)^4}\,dx = \frac{1}{3} < \infty$, so $h \in L^1$ and $\{h_n\} \subseteq L^1$

Slight modification before the end: use $\{g_n^*\}$ where $g_n^*(x) = g_n(x)\,\chi_{[0,n]}$ so that $\int_0^\infty g_n^*(x) = \int_0^n f(x/n)/(1+x)^4\,dx$. Note $\{g_n^*(x)\} \subseteq L^1([0,\infty))$ still, and $|g_n^*| \le |g_n| \le h_n$.

Lastly, by generalized DCT since $\{g_n^*\}, \{h_n\} \subseteq L^1$ and $|g_n^*| \le h_n$ $\forall n$, we have

$\lim \int_0^n \frac{f(x/n)}{(1+x)^4} = \lim \int_0^\infty g_n\cdot\chi_{[0,n]} = \lim \int_0^\infty g_n^* = \int_0^\infty \lim g_n^*$

$= \int_0^\infty \lim \frac{f(x/n)}{(1+x)^4}\chi_{[0,n]} = \int_0^\infty \frac{f(0)}{(1+x)^4} = f(0)/3 < \infty$ ✓

: $\int_0^\infty \frac{1}{(1+x)^4}\,dx = \int_1^\infty \frac{1}{u^4}\,du = \frac{-1}{3u^3}\Big|_1^\infty = 0 - \frac{-1}{3} = \frac{1}{3}$

$u = 1+x \quad du = dx$

FALL 2018

1. **Write the definition** of a separable metric space, then prove the following three statements:

   (a) **Show** that every infinite compact metric space (K, d) is separable.

   (b) Consider the metric space (X, d), where X is the set of bounded sequences of real numbers, i.e of $\mathbf{x} = \{x_n\}_1^\infty$, $x_n \in \mathbb{R}$, equipped with the distance
   $$d(\mathbf{x}, \mathbf{y}) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$
   (this space is known as ℓ^∞). **Show** that (X, d) is *not* separable.

   (c) **Show** that if (X, d) is a separable metric space then the cardinality of X can *not* be larger than the cardinality of $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$ (or in other words the cardinality of $\mathbb{R}$).

a metric space is separable if it has a countable, dense subset.

a.) can also show a countable base ($\Rightarrow$ separable)
compact $\Rightarrow$ every open cover has a finite subcover
Open cover: $\{B_{1/n}(x) \mid x \in X\}$
Then $\exists \{x_{1,n}, \dots, x_{m_n,n}\}$ s.t. $\{B_{1/n}(x_{i,n}) \mid i \in [1, m_n]\}$ covers X
Define $A = \bigcup_{n \in \mathbb{N}} \bigcup_{i=1}^{m_n} x_{i,n}$
Then A is countable. Remains to show dense

every pt either in A or a limit pt of A $\hookrightarrow$ arbitrarily close

Let $\varepsilon > 0$. For any $x \in X$, consider $B_\varepsilon(x)$. Then for any n s.t. $\frac{1}{n} < \varepsilon$, $\exists\, x_{i,n}$ where $x_{i,n} \in B_\varepsilon(X)$. Otherwise $\{B_{1/n}(x_{i,n})\}$ couldn't form a cover of X since $x \notin B_{1/n}(x_{i,n})$ $\forall i$. Thus, $\exists\, x_{i,n} \in B_\varepsilon(x)$, so $\exists$ an element of A "arbitrarily close" to any $x \in X$

b.) For every $I \subset \mathbb{N}$, define $x_I \in X$ (seq $\{x_n\}$) as $x_{I,n} = \begin{cases} 1 & \text{if } n \in I \\ 0 & \text{if } n \notin I \end{cases}$

pts are the seq. dist btw any 2 nonequal seq is 1 in X so take $r = \frac{1}{2}$

Whenever $I \neq J$, get some seq. entry different btwn x_I & x_J, so $\sup_{n \in \mathbb{N}} |x_{I,n} - x_{J,n}| = |1 - 0| = 1$
Then $\{B_{1/2}(x_I) \mid I \subset \mathbb{N}\}$ is an $\underbrace{\text{uncountably } \infty}_{2^{\mathbb{N}} \text{ possible subsets } I}$ set of disjoint open balls in X.
Let A be any dense subset of X. Each $B_{1/2}(x_I)$ must contain some $a \in A$. Each $B_{1/2}(x_I)$ distinct, so each $a \in A$ distinct (call it $a_I \in B_{1/2}(x_I)$.) But this means A is uncountable as $2^{\mathbb{N}} = \#$ balls $= \#$ pts in A
So there cannot be a countable dense subset in this case, + thus X not separable

c.) X is separable, so $\exists$ a countable, dense subset A of X.
Every $x \in X$ is a limit pt of A, and each sequence of the countable pts in A has at most one limit pt. Then $|X| \leq \#$ seq. of pts in A.
Since there are countably many points in A, the $\#$ of seq is $= \mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$
Hence $|X| = 2^{\mathbb{N}}$

2. Let f be an integrable real-valued function on $\mathbb{R}$. Define a function $g : \mathbb{R} \mapsto \mathbb{R}$ by letting

$$g(x) = \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2}\, dy\,.$$

(a) Prove that g is continuous.

(b) Prove that g is continuously differentiable.*

(* Help for 2(b): State a theorem **without proof** which lists suitable conditions that allow you to "differentiate under the integral sign," then verify these conditions for this problem.)

a.) Let $\varepsilon > 0$.

$$|g(x_1) - g(x_2)| = \left| \int_{-\infty}^{\infty} f(y) e^{-(x_1-y)^2} dy - \int_{-\infty}^{\infty} f(y) e^{-(x_2-y)^2} dy \right|$$

$$= \left| \int_{-\infty}^{\infty} f(y) \left(e^{-(x_1-y)^2} - e^{-(x_2-y)^2}\right) dy \right|$$

Let $z_1 = x_1 - y$, $z_2 = x_2 - y$. Since e^{z^2} is continuous, $\exists\, \delta > 0$ s.t. whenever $|z_1 - z_2| < \delta$, $|e^{-z_1^2} - e^{-z_2^2}| < \varepsilon$.

So, we let $|x_1 - x_2| < \delta$ (using this same δ) and we get:

$|x_1 - x_2| = |x_1 - y + y - x_2| = |(x_1 - y) - (x_2 - y)| = |z_1 - z_2| < \delta$

$$\Rightarrow \left| \int_{-\infty}^{\infty} f(y)(e^{-z_1^2} - e^{-z_2^2})\, dy \right| \le \int_{-\infty}^{\infty} |f(y)| \cdot \underbrace{|e^{-z_1^2} - e^{-z_2^2}|}_{<\varepsilon} dy = \varepsilon \underbrace{\int_{-\infty}^{\infty} |f(y)|\, dy}_{<\infty \text{ since } f \text{ is integrable.}} = C\varepsilon \ \checkmark$$

Say $\|f\|_{L^1} = C < \infty$

b.) Folland Thm. 2.27b:

$g'(x) = \int \frac{\partial}{\partial x} f(x,y)\, dy$ if: ← shows that g is continuously differentiable

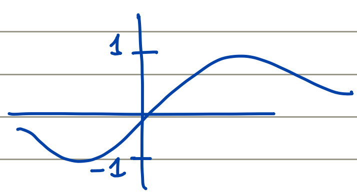
i.) $\partial f / \partial x$ exists

ii.) $\exists\, h \in L^1$ s.t. $|\frac{\partial}{\partial x} f(x,y)| < h(x)$ $\forall x, y$

i.) $\frac{\partial}{\partial x}\left(f(y) e^{-(x-y)^2}\right)$ exists?

$\frac{\partial}{\partial x}\left(f(y) e^{-(x-y)^2}\right) = f(y) \cdot (-2x + 2y) e^{-(x-y)^2}$ exists ✓

ii.) Plot $(-2x + 2y) e^{-(x-y)^2}$



Since $|(-2x + 2y) e^{-(x-y)^2}| \le 1$,

$|f(y) \cdot (-2x + 2y) e^{-(x-y)^2}| \le |f(y)| \in L^1$ (given)

Then the axioms of Thm 2.27b are satisfied, so $g(x)$ is continuously differentiable

3. Let $\pi(x, y) = x$ denote the projection of $\mathbb{R}^2$ onto $\mathbb{R}$, and let $\pi(A)$ denote the image under π of a subset A of $\mathbb{R}^2$.

(a) Let μ^* be an outer measure on the subsets of $\mathbb{R}$. Show that $\nu^*(A) := \mu^*(\pi(A))$ is an outer measure on the subsets of $\mathbb{R}^2$.

(b) Let λ^* be Lebesgue outer measure on the subsets of $\mathbb{R}$, and let $\rho^*(A) = \lambda^*(\pi(A))$. Show that if $A = B \times \mathbb{R}$, where B is a Lebesgue measurable subset of $\mathbb{R}$, then A is a ρ^* measurable set. Show where the assumption that A has this particular form is used.

outer measure has:

i.) $\mu^*(\emptyset) = 0$

ii.) $\mu^*(A) \leq \mu^*(B)$ when $A \subseteq B$

iii.) $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

a.) i.) $\nu^*(\emptyset) = \mu^*(\pi(\emptyset)) = \mu^*(\emptyset) = 0$ ✓

ii.) $\nu^*(A) = \mu^*(\underbrace{\pi(A)}) \leq \mu^*(\pi(B)) = \nu^*(B)$ ✓

$\pi(A) \subseteq \pi(B)$ in $\mathbb{R}$ when $A \subseteq B$ in $\mathbb{R}^2$

μ^* is an outer measure, so $\mu^*(\pi(A)) \leq \mu^*(\pi(B))$

iii.) $\nu^*(\bigcup_{i=1}^{\infty} A_i) = \mu^*(\pi(\bigcup_{i=1}^{\infty} A_i)) = \mu^*(\bigcup_{i=1}^{\infty} \pi(A_i)) \leq \sum_{i=1}^{\infty} \mu^*(\pi(A_i)) = \sum_{i=1}^{\infty} \nu^*(A_i)$ ✓

For a proj map, $\pi(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} \pi(A_i)$

b.) A is ρ^*-measurable iff $\rho^*(E) = \rho^*(E \cap A) + \rho^*(E \cap A^c)$ $\forall E \subset \mathcal{M}$

$\rho^*(A) = \lambda^*(\pi(A))$, $A = B \times \mathbb{R}$ for B Leb. msable

Then $\pi(A) = \pi(B \times \mathbb{R}) = B$ and $\pi(A^c) = \pi(B^c \times \emptyset) = B^c$

$\rho^*(E \cap A) = \lambda^*(\pi(E \cap A)) = \lambda^*(\pi(E) \cap \pi(A)) = \lambda^*(\pi(E) \cap B)$

$\rho^*(E \cap A^c) = \lambda^*(\pi(E \cap A^c)) = \lambda^*(\pi(E) \cap \pi(A^c)) = \lambda^*(\pi(E) \cap B^c)$

$\Rightarrow \rho^*(E \cap A) + \rho^*(E \cap A^c) = \lambda^*(\pi(E) \cap B) + \lambda^*(\pi(E) \cap B^c) = \lambda^*(\pi(E)) = \rho^*(E)$ ✓

substitute each term

Since B is λ^*-msable (given)

by def of ρ^*

4. Let $\Omega := [0,1]$ with Lebesgue measure.

(a) Let $f_n(x) = \cos 2\pi n x$. **Show** that $f_n \to 0$ weakly in $L^2(\Omega)$, but f_n does not converge to 0 a.e. in Ω or in measure.

(b) Let $f_n(x) = n\chi_{(0,1/n)}$. **Show** that $f_n \to 0$ a.e. and in measure, but f_n does not converge to 0 weakly in $L^p(\Omega)$ for any $p \geqslant 1$.

Recall: *A sequence of $\{g_n\}$ of L^p-functions converges to g weakly in $L^p(\Omega)$ if $\int_\Omega g_n\varphi \to \int_\Omega g\varphi$ for every $\varphi \in L^q(\Omega)$ (the dual of $L^p(\Omega)$) with $1/p + 1/q = 1$.*

Note: Bounded + ptwise conv $\Rightarrow$ conv in L^2 by DCT

L^2 conv $\Rightarrow$ ptwise conv subseq.

(Holder: $= \|f\varphi\|_{L^1} \le \|f\|_2 \|\varphi\|_2$)

a.) In L^2, $p = q = 2$. Let $\varphi \in L^2(\Omega)$, then

WTS: $=0$

$|\int f_n\varphi - \int f\varphi| = |\int \cos 2\pi n x\,\varphi - \int 0\varphi| = |\int \cos 2\pi n x\,\varphi| \le \int |f\varphi|$

$\le \|f\|_2 \cdot \|\varphi\|_2 = |\int_0^1 \cos^2 2\pi n x|^{1/2} \cdot \|\varphi\|_{L^2} = 0 \cdot \underbrace{\|\varphi\|_{L^2}}_{<\infty} = 0$ ✓

$= \left|\frac{1}{2\pi n} \sin 2\pi n x \Big|_{x=0}^{1}\right|^{1/2} = \left|\frac{1}{2\pi n}(\overset{=0}{\sin 2\pi n} - \overset{=0}{\sin 0})\right|^{1/2} = 0$

Then $f_n \to 0$ weekly in L^2

Check conv. a.e.: For any $x \in [0,1]$, $\varepsilon > 0$, $\exists N$ s.t. whenever $n \ge N$, $|f_n(x) - f(x)| < \varepsilon$.

$|f_n(x) - f(x)| = |\cos 2\pi n x - 0| = |\cos 2\pi n x|$

Counterex: let $x = \frac{1}{2}$, $\varepsilon < 1$. Identify N s.t. $\forall n \ge N$, $|\cos 2\pi n \cdot \frac{1}{2}| < \varepsilon < 1$

$\Rightarrow \forall n \ge N$, $|\cos \pi n| < 1$. Note for $n \in \mathbb{N}$, $\cos(\pi n) = \pm 1$. Then $|\cos \pi n| = 1$ for any n, so no matter what N we choose, $|\cos \pi n| < 1$ cannot happen ↯

So $f_n \not\to f$ ptwise

Since $\Omega = [0,1]$ is a finite m.s., conv. in $\mu \Rightarrow$ conv. ptwise.

By contrapos., since we showed that $f_n \not\to f$ a.e., then $f_n \overset{\mu}{\not\to} f$.

b.) $f_n(x) = n\chi_{(0,1/n)}$ WTS: $f_n \to 0$ a.e. + in μ

Let $\varepsilon > 0$, and for $x \in [0,1]$ choose N s.t. $x > 1/N$. Then whenever $n \ge N$, $\frac{1}{N} \ge \frac{1}{n}$, so $x > 1/n$, and

$|f_n(x) - f(x)| = |n\chi_{(0,1/n)}(x) - 0| = |n\underbrace{\chi_{(0,1/n)}(x)}_{=0 \text{ as } x > 1/n}| = 0 < \varepsilon$ ✓

Then $f_n \to 0$ a.e.

To show conv in μ, WTS: $\mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) \to 0$

$= \mu(\{x \mid |n\chi_{(0,1/n)}| > \varepsilon\}) \to 0$

Let $\varepsilon > 0$, and $\varepsilon < 1$, then $\{x \mid |n\chi_{(0,1/n)}| > \varepsilon\} = x \in (0, \frac{1}{n})$. As $n \to \infty$, $\mu((0,\frac{1}{n})) \to \mu(0) = 0$

so $f_n \overset{\mu}{\to} 0$

However, f_n does not converge weekly to 0 in L^p for any $p \ge 1$

To converge weakly, need $\int f_n\varphi \to \int f\varphi$ for any $\varphi \in L^q$ (cr: Leonidas)

Counterex: $\varphi = 1$ ($\in L^q$ $\forall q \ge 1$)

$\int f_n\varphi = \int_0^1 n\chi_{(0,\frac{1}{n})} \cdot 1\,dx = \int_0^{1/n} n\,dx = n(\frac{1}{n} - 0) = 1$

But $\underset{\ne 1}{\int f\varphi = 0}$ so does not converge weakley

5. Let f be an integrable real-valued function on $\mathbb{R}$. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \mapsto \mathbb{R}$ be a nonnegative measurable function. The hypograph of f is the subset $HG(f)$ of $X \times \mathbb{R}$ defined by

$$HG(f) = \Big\{(x,t) \in X \times \mathbb{R} : 0 \leqslant t \leqslant f(x)\Big\}.$$

(That is, $HG(f)$ is the "region under the graph of f".)

(a) **Prove** that the set $HG(f)$ is $\mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$-measurable, where $\mathcal{B}_{\mathbb{R}}$ is the σ-algebra of Borel subsets of $\mathbb{R}$, and $\mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$ is the product of the σ-algebras $\mathcal{A}$ and $\mathcal{B}_{\mathbb{R}}$, i.e., the σ-algebra of subsets of $X \times \mathbb{R}$ generated by the products $A \times B$, $A \in \mathcal{A}$, $B \in \mathcal{B}_{\mathbb{R}}$.

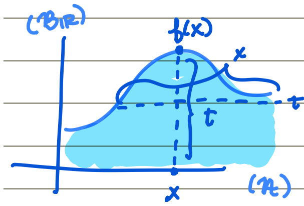
(b) Let $h_f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be the function given by

$\mathcal{B}_{\mathbb{R}}$ $\mathcal{B}_{\mathbb{R}}$

$$h_f(t) = \mu\Big(\{x \in X : f(x) \geqslant t\}\Big),$$

where $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geqslant 0\}$. **Prove** that h_f is Borel measurable and that

$$\int_X f\,d\mu = \int_0^\infty h_f(t)\,dt.$$



$HG(f) \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$
$HG(f) = \int h_f(t)\,dt$

a.) $HG(f)$ is $\mathcal{A}\otimes\mathcal{B}_{\mathbb{R}}$-msable $\iff$ $HG(f) \in \mathcal{A}\otimes\mathcal{B}_{\mathbb{R}}$ (gen by gen sets from each of $\mathcal{A}, \mathcal{B}_{\mathbb{R}}$) WTS:

$HG(f)_x = \{t \in \mathbb{R} \mid t \in [0, f(x)]\}$ $\rightsquigarrow$ generated by closed intervals of form $[0,b]$, which also generate $\mathcal{B}_{\mathbb{R}}$, so for each $x \in X$, $HG(f)_x \in \mathcal{B}_{\mathbb{R}}$

$HG(f)_t = \{x \in X \mid f(x) \geq t\}$

$f: X \to \mathbb{R}$ is msable, so $f^{-1}(E)$ is $\mathcal{A}$-msable for any $E \in \mathcal{B}_{\mathbb{R}}$

$f(HG(f)_t) = \{f(x) \mid f(x) \geq t\} = [t, \sup_x f(x)]$ } of form $[a,b]$, and $[a,b] \in \mathcal{B}_{\mathbb{R}}$

Then $f^{-1}(\underbrace{[t, \sup_x f(x)]}_{\text{msable}}) = \underbrace{HG(f)_t}_{\Rightarrow \text{ msable}} \in \mathcal{A}$ for each $t \in \mathbb{R}$

Then $HG(f) \in \mathcal{A}\otimes\mathcal{B}_{\mathbb{R}}$, so is $\mathcal{A}\otimes\mathcal{B}_{\mathbb{R}}$-msable

b.) h_f Borel msable iff $h_f^{-1}(E) \in \mathcal{B}_{\mathbb{R}}$ whenever $E \in \mathcal{B}_{\mathbb{R}}$

let $E \in \mathcal{B}_{\mathbb{R}}$. For each $e \in E$, $h_f^{-1}(e) = \{t \mid \mu(\{x \in X \mid f(x) \geq t\}) = e\}$

so, $h_f^{-1}(E) = \{t \mid \mu(\{x \in X \mid f(x) \geq t\}) \in E\}$

$\boxed{\text{Mark:}}$ $h_f(t)$ is nonincreasing (as t increases, $h_f(t)$ const. or dec.)

Then the preimage of $h_f(t)$ is an interval!

$\hookrightarrow$ Intervals are in $\mathcal{B}_{\mathbb{R}}$, so preimages of msable sets are msable, hence $h_f(t)$ is Leb. msable

$HG(f) \in \mathcal{A}\otimes\mathcal{B}_{\mathbb{R}}$, so $(\mu_x \times \mu_t)(HG(f)) = \int_{\mathbb{R}}\int_X \chi_{HG(f)}\, d\mu(x)\, d\mu(t)$ — nonneg., intable

$= \int_X\int_{\mathbb{R}} \chi_{HG(f)}\, d\mu(t)\, d\mu(x)$

$= \int_X\int_{\mathbb{R}} \chi_{HG(f)_x}\, d\mu(t)\, d\mu(x)$

$= \int_X\int_{\mathbb{R}} \chi_{\{t \in \mathbb{R} \mid t \in [0, f(x)]\}}\, d\mu(t)\, d\mu(x)$

$= \int_X\int_0^{f(x)} 1\, d\mu(t)\, d\mu(x)$

$= \int_X f(x)\, d\mu(x)$

$= \int_{\mathbb{R}}\int_X \chi_{HG(f)_t}\, d\mu(x)\, d\mu(t)$

$= \int_{\mathbb{R}}\int_X \chi_{\{f(x) \geq t > 0\}}\, d\mu(x)\, d\mu(t)$

$= \int_{\mathbb{R}} \mu(\{x \mid f(x) \geq t > 0\})\, d\mu(t)$

$= \int_0^\infty h_f(t)\, d\mu(t)$

$\int_{\mathbb{R}} = \int_0^\infty$ since $t > 0$ sign never enters $\int_X$, so assume int over only pos x (can shift up if not) $\Rightarrow$ use Fubini!

$\Rightarrow \int_X f(x)\, d\mu(x) = \int_0^\infty h_f(t)\, d\mu(t)$ ✓

SPRING 2019

1. Let $f \in L^1(\mathbb{R})$. We define the function g by

$$g(\alpha) = \int_{-\infty}^{\infty} f(x-\alpha)\frac{dx}{1+x^2}.$$

Prove that for all such f the function g is continuous. You must state carefully (but without proofs) all theorems you use.

(secretly abs cont.)

* Translation is continuous in L^p } similar to a S24 problem
↳ $\lim_{h\to 0} \|g(x+h) - g(x)\|_{L^1} = 0$
Do this by density of $C_c(\mathbb{R})$

$f \in L^1 \Rightarrow \forall \alpha,\ f(x-\alpha) \in L^1$
$\lim_{h\to 0} \|f(x-\alpha) - f(x-\alpha-h)\|_{L^1} = 0$ by (*)

Bound g:

$$|g(\alpha+h) - g(\alpha)| = \left|\int_{\mathbb{R}} f(x-(\alpha+h))\frac{dx}{1+x^2} - \int_{\mathbb{R}} f(x-\alpha)\frac{dx}{1+x^2}\right|$$

$$= \left|\int_{\mathbb{R}} (f(x-\alpha-h) - f(x-\alpha))\frac{dx}{1+x^2}\right| \le \left|\int_{\mathbb{R}} f(x-\alpha-h) - f(x-\alpha)\,dx\right|$$

$$\le \|f(x-\alpha-h) - f(x-\alpha)\|_{L^1} \to 0 \quad ✓$$

$\frac{1}{1+x^2} \le 1 \ \forall x \in \mathbb{R}$

2. Let

$$\tilde{\chi}_{[-1,1]}(x) = \begin{cases} 1, & x \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

and let I be a measurable subset of $\mathbb{R}$. We define

$$I(x) = \int_I \frac{\tilde{\chi}(x-y)}{1+y^2}\,dy. \quad \longrightarrow = 1 \text{ when } (x-y) \in [-1,1]$$

$$\left.\begin{array}{l} x-y \geq -1 \\ y \leq x+1 \\ x-y \leq 1 \\ y \geq x-1 \end{array}\right\} y \in [x-1, x+1]$$

(a) Prove that $I(x)$ is a nonnegative bounded L^1 function on $\mathbb{R}$.

(b) For $n \geq 1$ we define

$$a_n = \int_{-\infty}^{\infty} \cos(n^2 x) I(x)\,dx.$$

Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

$$\int_I \frac{\tilde{\chi}(x-y)}{1+y^2}\,dy = \int_{I \cap [x-1,x+1]} \frac{1}{1+y^2}\,dy \leq \int_I \frac{1}{1+y^2}\,dy$$

a.) WTS: $I(x) \in L^1$ nonneg + bdd $\Rightarrow \int_I |I(x)|\,dx < \infty$

$\Rightarrow \int_I \left| \int_I \frac{\tilde{\chi}(x-y)}{1+y^2}\,dy \right| dx$

$y^2 \geq 0 \Rightarrow 1+y^2 \geq 0$

$\tilde{\chi}(x-y) \geq 0$ so $\frac{\tilde{\chi}(x-y)}{1+y^2} \geq 0 \Rightarrow \int \frac{\tilde{\chi}(x-y)}{1+y^2} \geq 0$ so $I(x)$ is nonneg. ✓

$\frac{\tilde{\chi}(x-y)}{1+y^2} \leq \tilde{\chi}(x-y) \leq 1$ so $I(x) = \int_I \frac{\tilde{\chi}(x-y)}{1+y^2}\,dy \leq \int_I \tilde{\chi}(x-y)\,dy \leq \int_I 1\,dy = \mu(I) < \infty$

since $y^2 \geq 0$, $1+y^2 \geq 1$; by def of $\tilde{\chi}$; as I is measurable

Then $I(x)$ is bounded ✓

$\int_I \left| \int_I \frac{\tilde{\chi}(x-y)}{1+y^2}\,dy \right| dx \leq \int_I \int_I \left| \frac{\tilde{\chi}(x-y)}{1+y^2} \right| dy\,dx = \int_I \int_I \frac{\tilde{\chi}(x-y)}{1+y^2}\,dy\,dx \leq \int_I \mu(I)\,dx = \mu(I)^2 < \infty$

so $I(x) \in L^1$ ✓

b.) $a_n = \int_{-\infty}^{\infty} \cos(n^2 x) I(x)\,dx$ WTS: $\sum_{n=1}^{\infty} a_n$ conv abs i.e. $\sum_{n=1}^{\infty} |a_n| < \infty$

$\int_{-\infty}^{\infty} \int_I \underbrace{\cos(n^2 x)}_{\in L^1} \underbrace{\frac{\tilde{\chi}(x-y)}{1+y^2}}_{\in L^1}\,dy\,dx$) by Tonelli swap integrals

$\int_I \int_{-\infty}^{\infty} \cos(n^2 x) \frac{\tilde{\chi}(x-y)}{1+y^2}\,dx\,dy = \int_I \int_{y-1}^{y+1} \cos(n^2 x) \frac{1}{1+y^2}\,dx\,dy = \int_I \frac{1}{1+y^2} \int_{y-1}^{y+1} \cos(n^2 x)\,dx\,dy$

$x - y = -1 \Rightarrow x = y-1$
$x - y = 1 \Rightarrow x = y+1$

$= \int_I \frac{1}{1+y^2} \cdot \left(\frac{1}{n^2} \sin(n^2 x) \Big|_{x=y-1}^{x=y+1} \right) dy = \int \underbrace{\frac{1}{1+y^2}}_{\leq 1} \frac{1}{n^2} \underbrace{(\underbrace{\sin(n^2(y+1))}_{\leq 1} - \underbrace{\sin(n^2(y-1))}_{\leq 1})}_{\leq 1}\,dy$

$\leq \int_I \frac{1}{n^2}\,dy = \frac{1}{n^2} \mu(I)$

$\Rightarrow a_n \leq \frac{1}{n^2} \mu(I)$ $\forall n$ so $|a_n| \leq \left| \frac{1}{n^2} \mu(I) \right| = \frac{1}{n^2} \mu(I)$

So $\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{<\infty} \underbrace{\mu(I)}_{<\infty} < \infty$ ✓ so sum conv. abs!

3. (a) Let S be a **compact** metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements in S which has a finite number of accumulation points. We denote these points by $y_1, \dots, y_k$. Prove that it is possible to split the index set $\mathbb{N}$ into a disjoint union of k sets $S_j, j = 1, \dots, k$ such that each sub-sequence $\{x_n, n \in S_j\}$ converges to y_j.

(b) Let S be **any** metric space and the sequence $\{x_n\}$ has infinite but countable number of accumulation points $y_1, y_2, \dots$. Prove that it is possible to split the index set $\mathbb{N}$ into a disjoint union of sets $S_j, j = 1, 2, \dots$ such that each sub-sequence $\{x_n, n \in S_j\}$ converges to y_j.

any open cover has a finite subcover

$S \cap (U \setminus \{y_i\}) \neq \emptyset$ $\forall$ nbhd U of y_i

a.) let $r = \min\{d(y_i, y_j) \mid \forall i \neq j\}$, and build a set of balls $B_{r/2}(y_i)$ for each $i \in [1, k]$. Note that by construction, each $B_{r/2}(y_i)$ is disjt.
each y_i is an accumulation pt, so $\{x_n\} \cap (B_\varepsilon(y_i) \setminus \{y_i\}) \neq \emptyset$ $\forall \varepsilon \le r/2$.
let $B = \bigcup_{i=1}^{k} B_{r/2}(y_i)$. Then $\exists N$ s.t. $\forall n \ge N$, $x_n \in B$
↳ assume not. Then in metric spaces, compactness = sequential compactness. So S is sequentially compact, and $\{x_n\}$ has a convergent subsequence.
If no such N exists, then there are infinitely many x_n's outside of B, so the limit of $\{x_{n_k}\}$ is outside of B as well. But this contradicts each y_k being an accumulation pt.
Thus, such an N must exist.

$S_i = \{x_{i,n}\}$

Then $\forall n \ge N$, if $x_n \in B_{r/2}(y_i)$, then let $x_n \in \{x_{i,n}\}$. Since each $B_{r/2}(y_i)$ is disjoint, each x_n is in exactly one $B_{r/2}(y_i)$, so each pt of $\{x_n\}$ belongs to exactly one subseq. (i.e. $\{x_n\}$ is partitioned $\forall n \ge N$). Then $\{x_{i,n}\} \to y_i$ since there are ∞ many pts of $\{x_n\}$ (and thus in S_i) arbitrarily close to y_i as y_i is an accumulation pt.
For the first N pts of $\{x_n\}$, partition them however you like as they won't affect the convergence of subseqs

b.) Repeat same process as part (a.), but now let $r = \inf\{d(y_i, y_j) \mid \forall j \ge i\}$
Since each y_i is an accumulation pt, there are ∞ many x_n in $B_\varepsilon(y_i)$ $\forall \varepsilon < r/2$, and so we can partition $\{x_n\}$'s into subseq for each $i \in \mathbb{N}$.
However, now that we are working in an arbitrary metric space w/o compactness, we lost seq. compactness, so we can't guarantee that there are only finitely many points outside of B to distribute.
This is okay! There are (at most) countably infinite pts in $S \setminus B$, so enumerate this seq. $\{x_{k+1,i}\}$. Then since there are countably infinite subseq, take $\{x_{k+1,i}\}$ and add it as the first term in the subseq. $\{x_{n_i}\}$. Then $x_{n_i} \to y_i$ still, and all points in $\{x_n\}$ have been partitioned properly!

↳ CR: DANAE

4. (a) Suppose that $a(x)$ is a bounded measurable function on $[0,1]$, and $u(x)$ is an absolutely continuous function on $[0,1]$, which satisfies $u'(x) = a(x)u(x)$ for *a.e.* $x \in [0,1]$. Further suppose that $u(0) = 0$. Prove that $u(x) \equiv 0$ on $[0,1]$.

(b) Provide an example to show that the statement of part (a) does not hold if the absolute continuity condition on $u(x)$ is weakened. Namely, exhibit a bounded measurable $a(x)$ on $[0,1]$, $u(x)$ continuous on $[0,1]$ with $u'(x)$ existing and satisfying $u'(x) = a(x)u(x)$ for *a.e.* $x \in [0,1]$, but $u(x) \not\equiv 0$ on $[0,1]$.

a.) FTC: $f(x)$ abs cont on $[a,b]$ $\iff f(x) = f(a) + {}_a\int^x g(t)\,dt$ for some $g \in L^1$ $\forall x \in [a,b]$
$\iff f(x) = f(a) + {}_a\int^x f'(t)\,dt$ so f diffable a.e. + $f' \in L^1$

By FTC, $u(x)$ abs cont so
$u(x) = u(0) - {}_0\int^x u'(t)\,dt$
$= 0 - {}_0\int^x a(t)\,u(t)\,dt$

Hint: let $f(y) = {}_0\int^y |u(t)|\,dt$ so $f'(y) = |u(y)|$

Then $|u(x)| = |{}_0\int^x a(t)u(t)\,dt|$
$\le {}_0\int^x \underbrace{|a(t)|}_{\le M \text{ as } a(t) \text{ is bdd}} \cdot |u(t)|\,dt$
$\le M\,{}_0\int^x |u(t)|\,dt$
$\le M f(x)$

$\Rightarrow f'(y) = |u(y)| \le M f(y)$

let $g(x) \le f(x)$ for a.e. x, and now we must solve the IVP:
$\hookrightarrow$ $f'(y) \le M f(y)$ $\quad f(0) = {}_0\int^0 |u(t)|\,dt = u(0) = 0$
$g'(y) = M g(y)$ $\quad g(0) = 0$

Two methods to solve the DE:

METHOD 1:

$g(x) = Ce^{Mx}$
$g(0) = 0 \Rightarrow C\underbrace{e^{M\cdot 0}}_{\ne 0} = 0$

Thus $C = 0$, so $g = 0 \cdot e^{Mx} = 0$

METHOD #2:

$e^{-Mx} f' \le e^{-Mx} Mf$
$e^{-Mx} f' - e^{-Mx} Mf \le 0$
$e^{-Mx}(f' - Mf) \le 0$
$\Rightarrow \frac{d}{dx}(e^{-Mx} f) \le 0$

$\forall x$, $f(x) \le g(x) = 0$
But f is nonneg ($f = \int |u(t)|\,dt$)
so $f_y \equiv 0$, and thus
$0 = {}_0\int^y |u(t)|\,dt \Rightarrow u \equiv 0$ ✓

b.) Need: $a(x)$ cont, bdd, msable on $[0,1]$
$u(x)$ cont, diffable
$u'(x) = u(x)\,a(x)$
But $u \not\equiv 0$

Try: Cantor function
$u(x)$ cont + diffable a.e. (constant a.e., but $u \not\equiv 0$)
letting $a \equiv 0$, $u'(x) = 0 \cdot u(x) = 0$ ✓
$\hookrightarrow$ cont.
Then the Cantor function satisfies the desired properties for $u(x)$

5. Assume that $\{f_n\}$ is a sequence of elements of $L^2(0,1)$ which satisfy

$$\sup_n \|f_n\|_{L^2(0,1)} < \infty.$$

Further assume that there exists a function $f : (0,1) \to \mathbb{R}$ with $f_n \to f$ a.e. in $(0,1)$.

$\mu(X) < \infty$ Egorov!

(a) Prove that $f \in L^2(0,1)$.

(b) Prove that for any $g \in L^2(0,1)$ there holds

$$\lim_{n\to+\infty} \int_0^1 f_n g\, dx = \int_0^1 f g\, dx.$$

a.) $\{f_n\} \subseteq L^2(0,1)$ w/ $\sup \|f_n\|_{L^2} < \infty$. $f_n \to f$ a.e.

WTS: $f \in L^2(0,1)$

$f_n \to f$ a.e., Then $|f_n| \to |f|$ a.e. and $|f_n|^2 \to |f|^2$

$\{|f_n|^2\} \subseteq L^+$, so use Fatou:

$\int \underbrace{\liminf |f_n|^2} \le \liminf \int |f_n|^2 \le \limsup \int |f_n|^2$

$= \limsup \|f_n\|_{L^2}^2$

$= \int |f|^2$ $\quad < \infty$ since $\sup \|f_n\|_{L^2} < \infty$

Then $\|f\|_{L^2}^2 < \infty$, so $f \in L^2(0,1)$ ✓

b.) Not necessarily monotone or dominated, so can't just pull in limit

Hint: Egorov

$f_n \to f$ a.e. on $(0,1)$, + $\mu((0,1)) = 1 < \infty$. By Egorov, $\forall \varepsilon > 0$, $\exists E \subseteq (0,1)$ s.t. $\mu(E) < \varepsilon$ + $f_n \rightrightarrows f$ on E^c. Then $\forall \varepsilon' > 0$, $\exists N$ s.t. $\forall n \ge N$, $|f_n(x) - f(x)| < \varepsilon'$ $\forall x \in E^c$.

WTS: ${}_0\int^1 f_n g \to {}_0\int^1 fg$, or $\left|{}_0\int^1 f_n g - {}_0\int^1 fg\right| = \left|{}_0\int^1 f_n g - fg\right| \to 0$

$\left|{}_0\int^1 (f_n - f)g\right| \le {}_0\int^1 \underbrace{|f_n - f| \cdot |g|}_{= \|(f_n - f)\cdot g\|_{L^1}} \underset{\text{Holder}}{\le} \|f_n - f\|_{L^2} \cdot \underbrace{\|g\|_{L^2}}_{<\infty} = \|g\|_{L^2} \cdot \left({}_0\int^1 |f_n - f|^2\right)^{1/2}$

$= \|g\|_{L^2} \cdot \Big[\underbrace{{}_E\int |f_n - f|^2}_{Ⓐ} + \underbrace{{}_{E^c}\int |f_n - f|^2}_{Ⓑ}\Big]^{1/2}$

By Egorov, $\forall n \ge N$, $|f_n(x) - f(x)| < \varepsilon'$ in E^c. Let $\varepsilon' = \sqrt{\varepsilon}$.

$\Rightarrow {}_{E^c}\int |f_n - f|^2 < {}_{E^c}\int (\varepsilon')^2 = \varepsilon \cdot \mu(E^c)$

Ⓑ $< \varepsilon^2 \cdot \mu(E^c)$

$\sup \|f_n - f\| < \infty$ + $f_n, f \in L^2$ $\Rightarrow$ $|f_n - f| \le C$ for some $C < \infty$

$\Rightarrow {}_E\int |f_n - f|^2 \le {}_E\int C^2 = C^2 \underbrace{\mu(E)}_{<\varepsilon \text{ by Egorov}} < C^2 \varepsilon$

Ⓐ $< C^2 \varepsilon$

Then $\underbrace{\|g\|_{L^2}}_{<\infty,\ \text{say} = D} (Ⓐ + Ⓑ)^{1/2} < D(\varepsilon^2 \cdot \underbrace{\mu(E^c)}_{\le 1 \text{ by monotonicity as } E^c \subset [0,1]} + C^2 \varepsilon)^{1/2} \le D(\varepsilon + C^2 \varepsilon)^{1/2} = \underbrace{D}_{<\infty}\sqrt{\varepsilon}\,\underbrace{(1 + C^2)^{1/2}}_{<\infty} < \varepsilon$ ✓

so $\left|{}_0\int^1 f_n g - fg\right| \to 0$ as $n \to \infty$ ✓

FALL 2019

↗ range doesn't include 0, so 1/f fine

1. If $f : [0,1] \to (0,\infty)$ is absolutely continuous, must $1/f$ be? Prove it if so, and exhibit a counterexample if not.

abs cont on $[0,1]$ $\Rightarrow$ For every $\varepsilon>0$, $\exists\, \delta>0$ s.t. whenever $\sum_{i=1}^N |b_i-a_i|<\delta$, then $\sum_{i=1}^N |f(b_i)-f(a_i)|<\varepsilon$

↳ for $(a_i,b_i)\subseteq[0,1]$, disjoint

Since f is cont on a cpt set, $\exists M>0$ s.t. $\forall x\in[0,1]$, $f(x)\le M$ (has an inf.)

$\forall x\in[0,1]$, $\left|\frac{1}{f(x)}-\frac{1}{f(y)}\right| = \left|\frac{f(y)-f(x)}{f(x)f(y)}\right| \le \frac{|f(y)-f(x)|}{M^2}$

Let $\varepsilon>0$. Then $\exists\,\delta>0$ + $\{(a_i,b_i)\}$ satis ↺. Then

$\sum_{i=1}^n \left|\frac{1}{f(b_i)}-\frac{1}{f(a_i)}\right| = \sum_{i=1}^n \left|\frac{f(a_i)}{f(b_i)\cdot f(a_i)}-\frac{f(b_i)}{f(a_i)\cdot f(b_i)}\right| = \sum_{i=1}^n \left|\frac{f(a_i)-f(b_i)}{f(b_i)\cdot f(a_i)}\right| \le \sum_{i=1}^n \frac{|f(a_i)-f(b_i)|}{M^2}$

$= \frac{n}{M^2}\sum_{i=1}^n |f(a_i)-f(b_i)| < \frac{n}{M^2}\varepsilon$ ✓

2. Let (X,Σ,μ) be a measure space and let $f\in L^1(X,\mu)$. Show that for every $\epsilon>0$, there exists $E\in\Sigma$ such that $\mu(E)<\infty$ and

$$\int_{X\setminus E}|f|\,d\mu<\epsilon.$$

Hint: you can define E using values of $f(x)$

$E_n=\{x \mid |f(x)|>1/n\}$

$f\in L^1(X,\mu) \Rightarrow \int_X|f| = \int_E|f| + \int_{X\setminus E}|f| <\infty$

$\Rightarrow \int_E|f|<\infty$

check that $\mu(E)<\infty$: $\int_E|f|\,dx \ge \int_E \frac{1}{n}\,dx = \frac{1}{n}\mu(E_n)$

If $\mu(E_n)=\infty$, then $\int_E|f|\,dx\ge\infty$, which contradicts $f\in L^1(X,\mu)$

Great! Then check last cond:

$E_n=\{x \mid |f(x)|>1/n\} \Rightarrow E_n^c=\{x \mid |f(x)|\le 1/n\}$

$\int_{X\setminus E}|f|\,dx = \int_{E_n^c}|f|\,dx \le \int_{E_n^c}1/n\,dx = 1/n\cdot \underbrace{\mu(E_n^c)}_{<\infty}$ as $n\to\infty$, $\int_{X\setminus E_n}|f|\,dx\to 0$

Then for any $\varepsilon>0$, pick n s.t. $\varepsilon > 1/n\cdot\mu(E_n^c)$, + our desired claim is satisfied!

$\to 0$ as $n\to\infty$, so such an n exists

a more rigorous proof w/ DCT:

$g_n=|f|\cdot\chi_{E_n^c}$. Then $g_n\ge 0$ $\forall n$, + $|g_n(x)|\le|f|$ ($f\in L^1 \Rightarrow |f|\in L^1$ ✓), so use DCT + note that

$\lim_{n\to\infty} g_n = \lim_{n\to\infty}|f|\chi_{E_n^c}=0$

$\int \lim g_n = \lim\int g_n$

$=\int 0 = 0$ $\quad = \lim\int|f|\chi_{E_n^c}$ $\quad\Rightarrow \int_{X\setminus E_n}|f|\to 0$, so can find an n for any $\varepsilon>0$

3. Consider the space $X = \mathbb{R}$ endowed with distance function $d(x,y) = |e^x - e^y|$.

(a) Prove that d is a metric.

(b) Prove that (X,d) is not complete.

a.) For a metric, we need:

1.) $d(x,x) = 0$

$d(x,x) = |e^x - e^x| = 0$ ✓

2.) $d(x,y) = d(y,x)$

$d(x,y) = |e^x - e^y| = |e^y - e^x| = d(y,x)$ ✓

3.) $d(x,z) \le d(x,y) + d(y,z)$

$d(x,z) = |e^x - e^z| = |e^x - e^y + e^y - e^z| \le |e^x - e^y| + |e^y - e^z| = d(x,y) + d(y,z)$ ✓

b.) complete = every Cauchy seq in X has a lim pt in X

suffices to find Cauchy seq. w/ lim pt outside of $X = \mathbb{R}$

Cauchy: $\forall \varepsilon > 0, \exists N$ s.t. $\forall n,m \ge N$ $d(x_n, x_m) < \varepsilon$

Find seq Cauchy in this metric (but not the Euclidean one)

Try: $x_n = -n$. Then $d(x_n, x_m) = |e^{-n} - e^{-m}| = |\frac{1}{e^n} - \frac{1}{e^m}|$

Let $\varepsilon > 0$, and set $N = \log(1/\varepsilon)$. Then $\{x_n\}$ is Cauchy ✓

$n,m \ge N \Rightarrow \frac{1}{e^n}, \frac{1}{e^m} \le \frac{1}{e^N}$

$\Rightarrow \frac{1}{e^n} - \frac{1}{e^m} \le \frac{1}{e^N}$

Need $\frac{1}{e^N} < \varepsilon$ to get $|\frac{1}{e^n} - \frac{1}{e^m}| < \varepsilon$

$\Rightarrow 1 < \varepsilon \cdot e^N$

$\Rightarrow 1/\varepsilon < e^N$

$\{x_n\}$ Cauchy, so $\exists x$ s.t. $\{x_n\} \to x$.

AFSOC $x \in X$, then $d(x_n, x) = |e^{-n} - e^x|$, so

$\lim d(x_n, x) = \lim |e^{-n} - e^x| = |0 - e^x| = \underbrace{e^x > 0}_{}$

const! $\to$ ↯

4. Evaluate the integral

$$\int_0^1 \int_y^1 x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) dx\, dy,$$

making sure to justify every step.

$x^{-3/2} \in L^+([0,1])$ ✓

For $\cos(\frac{\pi y}{2x}) \in L^+$, need $\cos(\frac{\pi y}{2x}) \ge 0$

$\Rightarrow \frac{\pi y}{2x} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\Rightarrow \frac{y}{x} \in [-1, 1]$

$\Rightarrow y \in [-x, x]$

or $\Rightarrow x \in (-\infty, -y] \cup \underbrace{[y, \infty)}_{}$

$[y,1] \subset [y, \infty)$ ✓

Then $\cos(\frac{\pi y}{2x}) \in L^+([y,1] \times [0,1])$

So $x^{-3/2} \cos(\frac{\pi y}{2x}) \in L^+([y,1] \times [0,1])$

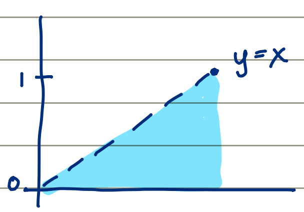
and $[0,1]$ is σ-finite, so we can use Tonelli!

By Tonelli:

$\int_0^1 \int_y^1 x^{-3/2} \cos(\frac{\pi y}{2x})\, dx\, dy = \int_0^1 \int_0^x x^{-3/2} \cos(\frac{\pi y}{2x})\, dy\, dx$

$\hookrightarrow [y,1] \times [0,1]$

$= [0,1] \times [0,x]$



$= \int_0^1 \left[x^{-3/2} \cdot \frac{2x}{\pi} \sin(\frac{\pi y}{2x}) \right]_0^x dx$

$= \int_0^1 x^{-3/2} \cdot \frac{2x}{\pi} \underbrace{\sin(\frac{\pi}{2})}_{=1} - 0 \; dx$

$= \frac{1}{\pi} \int_0^1 2x^{-1/2}\, dx$

$= \frac{1}{\pi} [4x^{1/2}]_0^1 = \frac{4}{\pi}(\sqrt{1} - \sqrt{0}) = \frac{4}{\pi}$ ✓

complete

5. Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

underograd real analysis proof

(a) Let $\{u_n\}_{n\geq 1}$ be a sequence in H so that $\sum_n \|u_n\| < \infty$. Prove that $\sum_n u_n$ converges in H. * partial sums form Cauchy seq. from abs conv test

(b) Suppose that $\|u_n\| \to \|u\|$ and $u_n \to u$ weakly, that is, $(u_n, v) \to (u, v)$, for all $v \in H$. Prove that $u_n \to u$ strongly, that is, $\|u_n - u\| \to 0$.

a.) Since $\Sigma_n \|u_n\|$ converges, given $\varepsilon > 0$ $\exists N$ s.t. if $n > m$ then $\Sigma_{k=m+1}^{n} \|u_k\| < \varepsilon$

Then $\|\Sigma_{k=m+1}^{n} u_k\| \leq \Sigma_{k=m+1}^{n} \|u_k\| < \varepsilon$

Δ ineq in Hil space

So $S_n = \Sigma_{k=1}^{n} x_k$ is Cauchy $\Rightarrow$ converges

$\boxed{\text{Fun Thm:}}$ X normed vector space, then X complete iff all abs conv seq converge.

↳ L^p spaces are complete!

b.) $\|u_n\| \to \|u\|$ & $\langle u_n, v\rangle \to \langle u, v\rangle$ $\forall v \in H$

<u>WTS:</u> $\|u_n - u\| \to 0$

$$\begin{aligned}
\|u_n - u\|^2 &= \langle u_n - u, u_n - u\rangle \\
&= \langle u_n, u_n - u\rangle - \langle u, u_n - u\rangle \\
&= \langle u_n, u_n\rangle - \langle u_n, u\rangle - \langle u, u_n\rangle + \langle u, u\rangle \\
&= \|u_n\|^2 + \|u\|^2 - \langle u_n, u\rangle - \langle u, u_n\rangle \\
&= \|u_n\|^2 + \|u\|^2 - \langle u_n, u\rangle - \overline{\langle u_n, u\rangle}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \|u_n - u\|^2 &= \lim_{n\to\infty} \left(\|u_n\|^2 + \|u\|^2 - \langle u_n, u\rangle - \overline{\langle u_n, u\rangle}\right) \\
&= \|u\|^2 + \|u\|^2 - \langle u, u\rangle - \overline{\langle u, u\rangle} \\
&= 2\|u\|^2 - 2\|u\|^2 \\
&= 0 \checkmark
\end{aligned}$$

SPRING 2020

1. (*a*) Exhibit a sequence of functions $f_n : [0,1] \to [0,1]$ so that $f_n \to 0$ in L^1 but for every $x \in [0,1]$, $\{f_n(x)\}$ has no limit.

(*b*) Must there exist a subsequence of f_n converging pointwise?

a.) Typewriter sequence

$f_1 = \chi_{[0,1]} = \chi_{E_1}$

$f_2 = \chi_{[0,\frac{1}{2}]} = \chi_{E_2}$

$f_3 = \chi_{[\frac{1}{2},1]} = \chi_{E_3}$

$f_4 = \chi_{[0,\frac{1}{3}]} = \chi_{E_4}$

$\vdots \qquad\qquad \vdots$

$\lim_{n\to\infty} \int |f_n - 0| \, d\mu = \lim_{n\to\infty} \int |\chi_{E_n}| \, d\mu = \lim_{n\to\infty} \mu(E_n) = 0$

so $f_n \to f$ in L^1

But for any $x \in [0,1]$, $x \in E_n$ for infinitely many $n \in \mathbb{N}$

so, $\nexists N$ s.t. $|f_n(x) - f(x)| < \varepsilon \ \forall n \geq N$ whenever $\varepsilon < 1$,

b.) $f_n \to f$ in L^1 $\Rightarrow f_n \xrightarrow{\mu} f \Rightarrow \exists$ subseq. of f_n which conv. ptwise to f

So yes!

2. State and prove Fatou's Lemma.

Fatou's Lemma: If $\{f_n\} \subseteq L^+$ then $\int \liminf f_n \leq \liminf \int f_n$

Proof: By MCT

Define $g_k := \inf_{n\geq k} f_n$, then g_k is monotone increasing and since $\{f_n\} \subseteq L^+$, $\{g_k\} \subseteq L^+$. Then by MCT:

$$\int \lim_{k\to\infty} g_k = \lim_{k\to\infty} \int g_k$$

$$\Rightarrow \int \lim_{k\to\infty} \inf_{n\geq k} f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n \leq \liminf \int f_n \ \checkmark$$

3. Suppose f is defined on $\mathbb{R}^2$ as follows: $f(x,y) = a_n$ if $n \leqslant x < n+1$ and $n \leqslant y < n+1$, $(n \geqslant 0)$; $f(x,y) = -a_n$ if $n \leqslant x < n+1$ and $n+1 \leqslant y < n+2$, $(n \geqslant 0)$; while $f(x,y) = 0$ elsewhere. Here $a_n = \sum_{k \leqslant n} b_k$, with $\{b_k\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s < \infty$.

$a_n \geq 0 \ \forall n$

(a) Verify that the slice $f_x(y) = f(x,y)$ is integrable, and show that for all x, $\int f_x(y)dy = 0$. Hence $\int \left(\int f(x,y)dy\right) dx = 0$.

(b) Prove that the slice $f^y(x) = f(x,y)$ is integrable and that $\int f^y(x)dx = a_0$ if $0 \leqslant y < 1$, and $\int f^y(x)dx = a_n - a_{n-1}$ if $n \leqslant y < n+1$ with $n \geqslant 1$. Hence the function $y \mapsto \int f^y(x)dx$ is integrable on $(0,\infty)$ and $\int \left(\int f(x,y)dx\right) dy = s$.

(c) Prove directly that $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy = \infty$.

$$f(x,y) = \begin{cases} a_n & n \leq x < n+1 \\ & n \leq y < n+1 \\ -a_n & n \leq x < n+1 \\ & n+1 \leq y < n+2 \\ 0 & \text{elsewhere} \end{cases}$$

a.) $f_x(y)$ fixes some x.

If $x < 0$, then $f(x,y) = 0 \ \forall y$, so $\int f_x(y)\,dy = 0$

If $x \geq 0$, then $x \in [n, n+1)$ for exactly one $n \geq 0$

Fix this n. Then $f(x,y) \neq 0$ iff $y \in [n,n+1)$ or $[n+1,n+2)$

$$\int f_x(y)\,dy = \int a_n \cdot \chi_{[n,n+1)} + -a_n \chi_{[n+1,n+2)}\,dy$$
$$= a_n \cdot m([n,n+1)) - a_n \cdot m([n+1,n+2))$$
$$= a_n \cdot 1 - a_n \cdot 1 = 0 \ \checkmark$$

Hence $\iint f(x,y)\,dy\,dx = \int \int f_x(y)\,dy\,dx = \int 0\,dx = 0$

Check integrable:

$$\int |f_x(y)|\,dy = \int |a_n \chi_{[n,n+1)}|\,dy + \int |-a_n \chi_{[n+1,n+2)}|\,dy$$
$$= a_n \cdot 1 + a_n \cdot 1 = 2a_n < \infty$$

So $f_x(y)$ integrable

b.) $f^y(x)$ fixes some y.

If $y < 0$, then $f(x,y) = 0 \ \forall y$, so $\int f^y(x)\,dx = 0$

If $y \in [0,1)$, then set $n=0$, so $f(x,y) \neq 0$ iff $x \in [0,1)$. Then

$\int f^y(x)\,dx = \int a_0 \chi_{([0,1))}\,dx = a_0 \cdot m([0,1)) = a_0 \ \checkmark$

setting $y \in [n+1,n+2)$

If $y \geq 1$, then $y \in [n,n+1)$ for exactly one n. Then $f(x,y) \neq 0$ iff $x \in [n,n+1)$ or $[n-1,n)$.

Then $\int f^y(x)\,dx = \int a_n \chi_{([n,n+1))} + -a_{n-1} \chi_{([n-1,n))}\,dx$
$$= a_n \cdot m([n,n+1)) - a_{n-1} m([n-1,n))$$
$$= a_n \cdot 1 - a_{n-1} \cdot 1 = a_n - a_{n-1} \ \checkmark$$

$\int |f^y(x)|\,dx =$

- $\hookrightarrow 0$ for $y < 0$
- $\hookrightarrow a_0$ for $y \in [0,1)$
- $\hookrightarrow \underbrace{a_n + a_{n-1}}_{<\infty}$ for $y \in [n,n+1)$, $n \geq 1$

so $f^y(x)$ is integrable for any $y \in \mathbb{R}$

Hence, $\iint f(x,y)\,dx\,dy = \int_0^\infty \int f^y(x)\,dx\,dy$

$$= \int_0^1 \int_0^\infty f^y(x)\,dx\,dy + \int_1^2 \int_0^\infty f^y(x)\,dx\,dy + \int_2^3 \int_0^\infty f^y(x)\,dx + \dots$$
$$= \int_0^1 a_0\,dy + \int_1^2 (a_1 - a_0)\,dy + \int_2^3 (a_2 - a_1)\,dy + \dots$$
$$= a_0 + (a_1 - a_0) + (a_2 - a_1) + \dots$$

Hint: Use MCT

$$= \sum_{n=0}^{\infty} \int_n^{n+1} \int_0^{\infty} f^y(x)\,dx\,dy$$
$$= \lim_{N\to\infty} \sum_{n=0}^{N} \int_n^{n+1} \int_0^{\infty} f^y(x)\,dx\,dy$$
$$= \lim_{N\to\infty} \left(\sum_{n=1}^{N} \int_n^{n+1} a_n - a_{n-1}\,dy\right) + \int_0^1 a_0\,dy$$
$$= \lim_{N\to\infty} a_0 + \sum_{n=1}^{N} a_n - a_{n-1}$$
$$= \lim_{N\to\infty} a_N = \lim_{N\to\infty} \sum_{n=1}^{N} b_n = \sum_{n=1}^{\infty} b_n = S \checkmark$$

↳ only thing that doesn't cancel

c.) $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|\,dx\,dy = \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x,y)|\,dx\,dy = \int_0^{\infty} \int_0^{\infty} |f^y(x)|\,dx\,dy$ → only nonzero here

$$= \int_0^1 \int_0^{\infty} |f^y(x)|\,dx\,dy + \int_1^2 \int_0^{\infty} |f^y(x)|\,dx\,dy + \dots$$

Use MCT again:

$$= \sum_{n=0}^{\infty} \int_n^{n+1} \int_0^{\infty} |f^y(x)|\,dx\,dy$$
$$= \lim_{N\to\infty} \sum_{n=0}^{N} \int_n^{n+1} \int_0^{\infty} |f^y(x)|\,dx\,dy$$
$$= \lim_{N\to\infty} a_0 + \sum_{n=1}^{N} \int_n^{N+1} \int_0^{\infty} |f^y(x)|\,dx\,dy$$
$$= \lim_{N\to\infty} a_0 + \sum_{n=1}^{N} a_n + a_{n-1} \quad \longrightarrow \quad = a_0 + (a_1 + a_0) + (a_2 + a_1) + \dots + (a_N + a_{N-1})$$
$$= \lim_{N\to\infty} a_N + \sum_{n=0}^{N-1} 2a_n$$

For any N, $a_N = \sum_{k=0}^{N} b_k = b_0 + \sum_{k=1}^{N} b_k \Rightarrow a_N \geq b_0$ for any N

positive since $\{b_k\}$ pos.

Then $\sum_{n=0}^{N} a_n \geq N \cdot b_0$. So

$$\lim_{N\to\infty} a_N + 2\sum_{n=0}^{N-1} a_n$$
$$\geq \lim_{N\to\infty} b_0 + 2(N-1)b_0$$
$$= \lim_{N\to\infty} 2Nb_0 \longrightarrow \infty \text{ as } N \to \infty$$

Then $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|\,dx\,dy \geq \infty \Rightarrow \int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|\,dx\,dy = \infty$ ✓

4. Let f be absolutely continuous on $[a, b]$. Is it true that the total variation function $TV(f_{[a,x]})$ is absolutely continuous? If so prove it, if not, provide a counterexample.

Could do this by showing $TV(f_{[a,x]}) = \int_a^x |f'(t)|\,dt$

Mark will do it w/ definitions:
Let $\varepsilon > 0$, then since f is abs cont on $[a,b]$, $\exists \delta > 0$ s.t. if $(a_1, b_1), \ldots, (a_N, b_N)$ disjt + $\sum_{i=1}^{N} |b_i - a_i| < \delta$, then $\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \varepsilon$

<u>Note:</u> let $x > y$ $TV(f_{[a,x]}) - TV(f_{[a,y]}) = TV(f_{[x,y]})$
For problems like this, take closed partitions + refine them

<u>WTS:</u> $TV(f_{[a_1,b_1]}) + \ldots + TV(f_{[a_N,b_N]}) < \varepsilon$

Let $P_i = \{a_i = c_{1,i}, \ldots, c_{N_i,i} = b_i\}$ be a partition of $[a_i, b_i]$
Pick P_i s.t. $\sum_{j=1}^{N_i - 1} |f(c_{j+1,i}) - f(c_{j,i})|$ is within ε/n of $TV(f_{[a_i,b_i]})$

$\Rightarrow TV(f_{[a_1,b_1]}) + \ldots + TV(f_{[a_n,b_n]}) \leq \sum_{i=1}^{n} \sum_{j=1}^{N_i - 1} |f(c_{j+1,i}) - f(c_{j,i})| + \varepsilon$

But we know that $\sum_{i=1}^{n} \sum_{j=1}^{N_i - 1} |(c_{j,i}, c_{j+1,i})| < \delta$

$\Rightarrow TV(f_{[a_1,b_1]}) + \ldots + TV(f_{[a_N,b_N]}) < 2\varepsilon$

<u>Strategy:</u> We broke up $[a_i, b_i]$ into a closed partition that is close to the TV, then we use the fact that f is abs cont.

5. Let $\tau(0) = \infty$, and for a non-zero integer $c \in \mathbb{Z}$, let $\tau(c)$ denote the smallest positive integer which does not divide c; so $\tau(12) = 5$ but $\tau(13) = 2$. Define the function $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_{\geq 0}$ by $d(a, b) = 2^{-\tau(a-b)}$ (of course $2^{-\infty} = 0$).

(*a*) Prove that $d(\cdot, \cdot)$ is a metric on the integers.

a.) $d(a,b) = 2^{-\tau(a-b)} = 2^{-\tau(b-a)} = d(b,a)$ ✓

$\hookrightarrow c \mid a \Leftrightarrow c \mid -a$

$d(a,b) \geq 0$. Suppose that $d(a,b) = 0 = 2^{-\tau(a-b)}$

$\Rightarrow \tau(a-b) = \infty$

$\Rightarrow a-b = 0$

If $a-b>0$ then $a-b+1 \nmid a-b$

Same if $a-b<0$

$d(a,c) \leq d(a,b) + d(a,c)$

$2^{-\tau(a-c)} \leq 2^{-\tau(a-b)} + 2^{-\tau(b-c)}$

Note that if $n \mid a-b$ + $n \mid b-c$, then $n \mid a-c$

$\Rightarrow \tau(a-c) \geq \min(\tau(a-b), \tau(b-c))$

$-\tau(a-c) \leq \max(\tau(a-b), \tau(b-c))$

$2^{-\tau(a-c)} \leq \max(-2^{-\tau(a-b)}, -2^{-\tau(b-c)})$

$\leq 2^{-\tau(a-b)} + 2^{-\tau(b-c)}$ ✓

(*b*) Describe explicitly the distance 2^{-n} ball about a point $a \in \mathbb{Z}$, that is

$$\{b \in \mathbb{Z} : d(a, b) < 2^{-n}\} = ?$$

[Hint: Let N_n denote the least common multiple of the numbers $1, \ldots, n$.]

b.) $\{b \in \mathbb{Z} \mid d(a,b) < 2^{-n}\}$

$= \{b \in \mathbb{Z} \mid 2^{-\tau(a-b)} < 2^{-n}\}$

$= \{b \in \mathbb{Z} \mid \tau(a-b) > n\}$

$= \{b \in \mathbb{Z} \mid j \mid (a-b) \ \forall j \in [1,n]\}$

$= \{b \in \mathbb{Z} \mid a-b \in N \cap \mathbb{Z}\}$

$= N \cap \mathbb{Z} + a$

for fixed a, all b s.t. $a-b$ divisible by all $j \leq n$, Then take all $a-b \in N_n$

$-b \in N_n - a$

$b \in a - N_n$

(*c*) From your answer in part (*b*), prove that the topology on $\mathbb{Z}$ induced by this metric is generated by the set of all non-constant arithmetic progressions, $q\mathbb{Z} + a = \{qm + a\}_{m \in \mathbb{Z}}$, with $q \geq 1$.

c.) Let $\mathcal{O} \subseteq \mathbb{Z}$ be open $\Rightarrow \forall x \in \mathcal{O}, \exists r_x$ s.t. $B_{r_x} \subseteq \mathcal{O}$

$\Rightarrow \mathcal{O} = \bigcup_{x \in \mathcal{O}} B_{r_x}(x)$

Fix x, $\exists n_x$ s.t. $2^{-n_x} < r_x$

$\Rightarrow B_{2^{-n_x}}(x) = N_{n_x}\mathbb{Z} + x \subseteq \mathcal{O}$

$\Rightarrow \mathcal{O} = \bigcup_{x \in \mathcal{O}} N_{n_x}\mathbb{Z} + x$

any open set $\mathcal{O}$ is a union of nonconst arithmetic progressions

left to check that these are non-const.

$q\mathbb{Z} + a$, $q \geq 1$ Let $nq + a \in q\mathbb{Z} + a$ and consider

$B_{2^{-N}}(nq+a) = \{b \in \mathbb{Z} \mid j \mid nq + a - b \ \forall j \in [1,n]\}$

$= N q\mathbb{Z} + nq + a \quad \subseteq q\mathbb{Z} + a$

$\Rightarrow$ Non const arithmetic progressions are both open + closed (*d*) Prove that the complement of an arithmetic progression is open (so arithmetic progressions are both open and closed).

d.) WLOG assume $a \geq 0$

$(q\mathbb{Z}+a)^c = \bigcup_{i=0}^{q-1} q\mathbb{Z}+i \quad i \neq a$

which is a union of open sets (from b.)

(*e*) Compute the complement of

$$\bigcup_{p \text{ prime}} (p\mathbb{Z}+0),$$

where the union runs over primes.

e.) $\forall n \in \mathbb{Z} \setminus \{\pm 1\}$ $\exists$ p prime s.t. $p \mid n$

additionally, $\pm 1 \notin p\mathbb{Z}$ for any p

$\Rightarrow \bigcup_{p \text{ prime}} (p\mathbb{Z}+0) = \mathbb{Z} \setminus \{\pm 1\}$

$\Rightarrow (\bigcup_{p \text{ prime}} (p\mathbb{Z}+0))^c = \{\pm 1\}$

(*f*) Conclude from (*e*) that there are infinitely many primes. [Hint: what is the cardinality of a non-empty open set?]

f.) If there are finitely many primes $\{p_1, \dots, p_n\}$

$\{\pm 1\} = (p_1\mathbb{Z}+0)^c \cap \dots \cap (p_n\mathbb{Z}+0)^c$

is a finite intersection of open sets.

But all open sets are infinite $\lightning$

FALL 2020

CR: SAM

1. Let E_k be a sequence of Lebesgue measurable subsets of $\mathbb{R}$. Let
$$E = \{x \in \mathbb{R} : x \in E_k \text{ for infinitely many } k\}.$$
   1. Show that E is Lebesgue measurable.
   2. Show that if $\sum_{i=1}^{\infty} |E_k| < \infty$, then $|E| = 0$.
   3. Assume instead only that $\lim_{k\to\infty} |E_k| = 0$. Must $|E| = 0$?

Can also do w/ Fatou by defining
$f_k = \chi_{A_k}$ where
$A_k := \bigcup_{i=1} E_i \setminus E_k$
(set difference transforms liminf $\longleftrightarrow$ limsup)

1.)

$E = \{x \mid x \in E_k$ for inf many $k\}$

For each x, $\exists N$ s.t. $\forall k \geq N$ $x \in E_k$

$E = \underbrace{\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} E_k}_{\in \mathcal{M} \text{ (σ-alg.)}}}_{\in \mathcal{M} \text{ (σ-alg.)}}$ $\Rightarrow E \in \mathcal{M}$ (measble)

} since σ-alg closed under ctable unions & intersects

E_k measble $\Rightarrow f_k = \chi_{E_k} \in L^+$

$E_k = \int \chi_{E_k}$

2.) If $\sum_{i=1}^{\infty} |E_i| < \infty$, then Borel-Cantelli says $|\limsup E_i| = 0$. But $\underbrace{\limsup E_i}_{\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty}} = E$, so $|E| = 0$

3.) $\lim_{k\to\infty} |E_k| = 0$. Does $|E| = 0$?

No, counterex: Typewriter seq.

$E_1 = [0,1]$

$E_2 = [0, \frac{1}{2}]$

$E_3 = [\frac{1}{2}, 1]$

$E_4 = [0, \frac{1}{3}]$

$E_5 = [\frac{1}{3}, \frac{2}{3}]$

$E_6 = [\frac{2}{3}, 1]$

$\vdots$

Then $\lim_{k\to\infty} |E_k| = 0$

But $E = \limsup E_k = [0,1]$

so $|E| = 1 \neq 0$

$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$
$= \bigcap_{n=1}^{\infty} [0,1]$
$= [0,1]$

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of integrable functions with $\int_{\mathbb{R}} |f_n| \leq 1$. Assume that there exists a measurable function f such that $|\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\}| \to 0$ as $n \to \infty$ for each $\epsilon > 0$.
conv. in μ

1. Show that there is a subsequence f_{n_k} which converges to f almost everywhere.

2. Show that f is integrable.

1.) WTS: conv in $\mu \Rightarrow \exists$ subseq conv a.e.
Hint: construct subseq w/ $\varepsilon_k = 1/k$, then $\mu = 1/2^k$

Define a sequence $\{\varepsilon_k\}$ w/ $\varepsilon_k = 1/k$. Then select f_{n_k} s.t. $\mu(\{x \mid |f_{n_k}(x) - f(x)| > 1/k\}) < 1/2^k$
pick something w/ convergent sum so we can apply Borel-Cantelli

AFSOC $f_{n_k} \not\to f$ a.e.
Then $\exists \varepsilon > 0$ s.t. $\nexists N$ s.t. $\forall n_k \geq N$, $|f_{n_k}(x) - f(x)| < \varepsilon$
OR $\exists \varepsilon > 0$ s.t. $|f_{n_k}(x) - f(x)| > \varepsilon$ for infinitely many n_k
i.e. $|f_{n_k}(x) - f(x)| > 1/k$ for infinitely many k

Let $E_k = \{x \mid |f_{n_k}(x) - f(x)| > 1/k\}$ and let $E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$
Then $\{x \mid f_{n_k}(x) \not\to f(x)\} \subseteq E$ (WTS: $\mu(E) = 0$)
Note that $\mu(E_k) < 1/2^k$, so $\sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$
Then by Borel-Cantelli: (sum converges)
$\mu(E) = \mu(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) = \mu(\limsup E_k) = 0$

Thus $f_{n_k} \to f$ a.e. ✓

2.) $\int |f_{n_k}| \leq 1$ $\forall n_k$, and $\{|f_{n_k}|\} \subseteq L^+$ since each f_{n_k} integrable
By Fatou:
$$\underbrace{\int \liminf |f_{n_k}|}_{= \int |f|} \leq \underbrace{\liminf \int |f_{n_k}|}_{= 1}$$
$\Rightarrow \int |f| \leq 1 < \infty$
so f is integrable ✓

3. Let M be a metric space, N be a complete metric space, and $S \subset M$ is a dense subset. Let $f : S \to N$ be a uniformly continuous function. Show that there exists a unique continuous function $g : M \to N$ such that $g|_S = f$.

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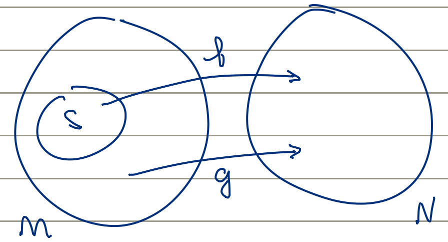
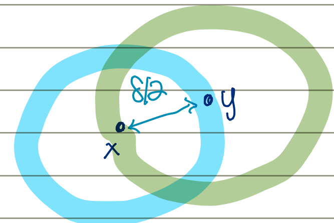




metric space!

S dense $\Rightarrow M = \overline{S}$. Then for any $x \in M$, $\exists \{x_n\} \subseteq S$ s.t. $\{x_n\} \to x$. Define $g(x) = \lim f(x_n)$. Then $g|_S = f$.
Since f is uni cont, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $d_M(a,b) < \delta$ for $a, b \in S$, then $d_N(f(a), f(b)) < \varepsilon$.
Since $\{x_n\} \to x$, the sequence is Cauchy in M, so $\forall \delta > 0, \exists L$ s.t. $\forall m, n \geq L$, $d_M(x_n, x_m) < \delta$.
This also means that $d_N(f(x_n), f(x_m)) < \varepsilon$. So $\{f(x_n)\}$ is also Cauchy, and since N is complete, $\{f(x_n)\} \to y$ for some $y \in N$.
Thus, $g(x) = \lim f(x_n) = y \in N$, so $g(x)$ exists $\forall x \in M$.

- g is well-defined

Let $\{x_n\}, \{y_n\} \subset M$ s.t. $x_n, y_n \to x$. Then the sequence defined by $x_1, y_1, x_2, y_2, x_3, y_3, \ldots$ also converges to x, and we know that $f(x_1), f(y_1), f(x_2), f(y_2), \ldots$ converges to $y \in N$. Then $f(x_n)$ and $f(y_n)$ are convergent subseq of this seq, so each must also converge to y.
Then $g(x) = \lim f(x_n) = \lim f(y_n) = y$ ✓

- g is continuous

Fix $x \in M$, let $\varepsilon > 0$, and pick the corresponding $\delta > 0$ using the uni cont of f. For any $y \in M$ s.t. $d_M(x,y) < \delta/3$, find sequences $\{x_n\}, \{y_n\} \subseteq S$ converging to x and y, resp. and within distance $\delta/3$ of x and y, resp. using the density of S in M. Then:

$$\begin{aligned} d_N(g(x), g(y)) &= d_N(\lim f(x_n), \lim f(y_n)) \\ &= \lim \underbrace{d_N(f(x_n), f(y_n))}_{<\delta} \\ &< \varepsilon \checkmark \end{aligned}$$

by uni cont of f

- g is unique

Let h be another such function. Let $x \in M$ and $\varepsilon > 0$, and let δ be the smaller of the δ's associated w/ ε under the continuity of g + h. Then pick $y \in S$ s.t. $d_M(x,y) < \delta$. Then

$$\begin{aligned} d_N(g(x), h(x)) &\leq d_N(g(x), f(y)) + d_N(f(y), h(x)) \\ &= d_N(g(x), g(y)) + d_N(h(y), h(x)) \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

Sending $\varepsilon \to 0$, we get that $d_N(g(x), h(x)) = 0$, so $g(x) = h(x)$ $\forall x \in M$ and so $g \equiv h$.

4. Construct a nondecreasing function $f:(0,1)\to\mathbb{R}$ whose discontinuity set is exactly $\mathbb{Q}\cap(0,1)$ (the rational numbers in $(0,1)$), or prove that such a function does not exist.

Note: true for any countable subset of $(0,1)$, not just $\mathbb{Q}$

If f nondec, $\Rightarrow$ discont. set is (at most) countable
↳ This problem is the converse

Since $\mathbb{Q}\cap(0,1)$ is countable, enumerate $\mathbb{Q}\cap(0,1)=\{q_1,q_2,\dots\}$
Let $a_n>0$ be a seq s.t. $\sum_{n=1}^{\infty}a_n<\infty$, let $f_n(x)=a_n\chi_{\{x\mid x\geq q_n\}}(x)$
ex: take $a_n=\frac{1}{n^2}$
Define $f(x)=\sum_{n=1}^{\infty}f_n(x)$
Since $\sum_{n=1}^{\infty}a_n<\infty$, then by absolute conv, we know $f(x):(0,1)\to\mathbb{R}$ is defined

For each n, f_n is cont on $(0,1)\setminus\mathbb{Q}$
By Weierstrass M-test,
$\sum_{n=1}^{N}f_n(x)$ conv uni to $f(x)$
Then on $(0,1)\setminus\mathbb{Q}$, $f(x)$ is cont.
(uniform limit of cont funcs is cont)

Weierstrass M-Test:
- $|f_n(x)|\leq M_n\ \forall x\in A$ ($\{M_n\}\geq 0$)
- $\sum_{n=1}^{\infty}M_n$ converges

$\Downarrow$

- $\sum_{n=1}^{\infty}f_n(x)$ conv abs + uni on A

Remains to show that $f(x)$ is discontinuous on $\mathbb{Q}$
Fix some q_n, then when $x<q_n$, $|f(x)-f(q_n)|\geq a_n>0$
when $x<q_n$ then $a_n\chi_{\{x\mid x\geq q_n\}}(x)=0$
$a_n\chi_{\{x\mid x\geq q_n\}}(q_n)=1$
Then $f(q_n)>f(x)$ for any $x<q_n$, so f has discontinuity at q_n for any n

5. Let $f(x,y)=\frac{xy}{(|x|+|y|)^{\alpha}}$ for $(x,y)\neq 0$, and $f(0,0)=0$, where $\alpha\in\mathbb{R}$. For what values of α is f integrable on $(-1,1)\times(-1,1)$? Justify your answer.

Notice that $\exists\, C_1,C_2>0$ s.t. $C_1(|x|^2+|y|^2)^{1/2}\leq|x|+|y|\leq C_2(|x|^2+|y|^2)^{1/2}$
↳ Idea: makes denom easier to work w/

also, $B_1\subseteq(-1,1)\times(-1,1)\subseteq B_2$
So, $\exists\,\tilde{C}_1,\tilde{C}_2>0$ s.t.

$$\tilde{C}_1\int_{B_1}\frac{|xy|}{(|x|^2+|y|^2)^{\alpha/2}}\leq\int_{(-1,1)\times(-1,1)}\frac{|xy|}{(|x|^2+|y|^2)^{\alpha/2}}\leq\tilde{C}_2\int_{B_2}\frac{|xy|}{(|x|^2+|y|^2)^{\alpha/2}}$$

Change to polar coords, it'll be easier to work there

$$\int_{B_R}\frac{|xy|}{(|x|^2+|y|^2)^{\alpha/2}}=\int_0^R\int_0^{2\pi}\frac{r^2|\sin\theta\cos\theta|}{r^{\alpha}}\cdot r\,dr\,d\theta=\underbrace{\underset{\substack{\uparrow\\ \text{const}>0}}{D}\int_0^R r^{3-\alpha}\,dr}_{\text{is integrable}\iff\alpha<4}$$

$x=r\cos\theta$
$y=r\sin\theta$
$(r^2)^{\alpha/2}=r^{\alpha}$

Combining the above estimates, see that $f(x,y)$ is integrable iff $\alpha<4$

SPRING 2021

1. Prove that a metric space (X, ρ) is complete if and only if for every decreasing sequence $F_1 \supseteq F_2 \supseteq \ldots$ of nonempty closed subsets of X with $\lim_{n\to\infty} \text{diam}(F_n) = 0$, the intersection $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$ for some $x_0 \in X$.
Hint: For any set $E \subseteq X$ we define its diameter by setting $\text{diam}(E) = \sup\{\rho(x,y) : x, y \in E\}$.

8

($\Rightarrow$) assume (X, ρ) complete. Then every Cauchy seq has lim pt in X.
Let $\{F_i\}$ be an decreasing seq. of sets as described.
Define seq of pts $\{x_n\}$ s.t. $x_i \in F_i$ for each i. (WTS: lim pt in intersect.)
Notice that $\forall \varepsilon > 0$, $\exists N$ s.t. $\text{diam}(F_n) < \varepsilon$ $\forall n \geq N$. Then whenever $m, n \geq N$, $\rho(x_n, x_m) < \varepsilon$. Thus, $\{x_n\}$ is a Cauchy seq.
By completeness of (X, ρ), $\exists x_0 \in X$ s.t. $\{x_n\} \to x_0$. Fix m, & $\forall n \geq m$ note that $\{x_n\}_{n\geq m} \subset F_m$. Since $\{x_n\}_{n\geq m}$ is a subseq of $\{x_n\}$, $\{x_n\}_{n\geq m}$ must also converge to x_0. Then since F_m is closed, F_m contains all lim. pts of seq. in F_m, so $x_0 \in F_m$, too. Then $x_0 \in \bigcap_{i=1}^{m} F_i$, and letting $m \to \infty$ we get $x_0 \in \bigcap_{i=1}^{\infty} F_i$ as desired ✓

($\Leftarrow$) assume for every $\{F_i\}$ as described, $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$ w/ $x_0 \in X$.
Let $\{x_n\}$ be a Cauchy seq in X (WTS: has lim pt in X).
AFSOC $\{x_n\}$ has no limit pt in X.
Define a seq of sets: $F_n := \{x_m \mid m \geq n\}$. Note that this is a decreasing seq of sets as (at most) a single pt is lost @ each step (& no pts added) and of course $F_n \supset F_{n+1}$ $\forall n \in \mathbb{N}$. Since $\{x_n\}$ has no limit pts, each F_n (vacuously) contains all its limit pts, so each F_n is closed.
Since $\{x_n\}$ is Cauchy, $\forall \varepsilon > 0$ $\exists N$ s.t. whenever $m, n \geq N$, $\rho(x_n, x_m) < \varepsilon$.
Then $\left.\begin{aligned}\text{diam}(F_n) &= \sup\{\rho(x,y) \mid x, y \in F_n\} \\ &= \sup\{\rho(x_j, x_k) \mid j, k \geq n\} < \varepsilon\end{aligned}\right\} \Rightarrow \text{diam}(F_n) \to 0$ ✓

Then $\{F_n\}$ satisfies the cond for our given property, so $\exists x_0 \in \bigcap_{n\in\mathbb{N}} F_n$. Since $x_0 \in F_n$ $\forall n \in \mathbb{N}$, then $\rho(x_m, x_0) < \varepsilon$ $\forall m \geq n \geq N$.
Thus $\{x_n\} \to x_0 \in \bigcap_{n\in\mathbb{N}} F_n \subseteq X$, so $\{x_n\}$ has a limit pt in X.
This is true for any Cauchy seq. in X, so (X, ρ) is complete ✓

2. Recall that a subset E of a metric space X is nowhere dense if $\operatorname{int}(\operatorname{cl}(E)) = \varnothing$.

(A) Is it true that every nowhere dense subset of $\mathbb{R}$ must have Lebesgue measure zero? Justify your answer.

(B) Give an example of a nowhere dense and uncountable subset of $\mathbb{R}$ which has Lebesgue measure zero.

(C) Is it true that every subset of the standard Cantor set $\mathcal{C}$ in $[0, 1]$ is Lebesgue measurable? Justify your answer.

a.) Not necessarily. Counterexample: consider the Fat Cantor set obtained by removing the middle 1/4-th from each interval at each successive step. Then:

$$m(C) = m([0,1]) - \sum_{k=1}^{\infty} \underbrace{\frac{1}{4^k}}_{\text{length removed @ step } k} \cdot \underbrace{2^{k-1}}_{\text{\# intervals removed @ step } k} = 1 - \sum_{k=1}^{\infty} \frac{2^{k-1}}{2^{2k}} = 1 - \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = 1 - \frac{1}{2} = \frac{1}{2} \neq 0$$

Note: $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$

$\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2}$

$\sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2}$

(Fat Cantor is nowhere dense for the same reasons that the regular Cantor set is.)

b.) Standard Cantor set.
Known to be nowhere dense, uncountable, + has $m(C) = 0$

c.) Yes. Lebesgue measure is a complete measure, so any subset of a null set (set w/ measure 0) is measurable. Then any subset of the Cantor set is measurable.

3. Let $(X, \mathcal{B}(X), \mu)$ and $(Y, \mathcal{B}(Y), \nu)$ be two σ-finite measure spaces and let $1 \leq p < \infty$. Show that for every nonnegative measurable function F on the product space $X \times Y$ with the product measure $\mu \times \nu$ we have

can use Fubini/Tonelli

$\in L^+$, and $F = |F|$

$$\left[\int_Y \left(\int_X F(x,y)d\mu(x)\right)^p d\nu(y)\right]^{1/p} \leq \int_X \left[\int_Y F(x,y)^p d\nu(y)\right]^{1/p} d\mu(x).$$

(*) Note that $p=1$ is simply Tonelli's theorem.
Consider the case of $p \geq 2$.

Hint: Observe that

$$\left(\int_X F(x,y)d\mu(x)\right)^p = \left(\int_X F(x,y)d\mu(x)\right)\left(\int_X F(x,y)d\mu(x)\right)^{p-1}$$

and apply Hölder's inequality.

$${}_y\int \left({}_x\int F(x,y)\, d\mu\right)^p d\nu = {}_y\int \left(\left({}_x\int F d\mu\right)\underbrace{\left({}_x\int F d\mu\right)^{p-1}}_{=g^{p-1} \in L^+ \text{ by Tonelli}}\right) d\nu$$

$$= {}_y\int \left({}_x\int F \cdot g^{p-1} d\mu\right) d\nu$$

By Tonelli: $= {}_x\int {}_y\int F g^{p-1} d\nu\, d\mu$

$$\leq {}_x\int \|F\|_{L^p} \cdot \|g^{p-1}\|_{L^{\frac{p}{p-1}}}\, d\mu$$ By Holder

Holder: ${}_y\int F g^{p-1} d\nu = \|F g^{p-1}\|_{L^1} \leq \|F\|_{L^p} \cdot \|g^{p-1}\|_{L^{\frac{p}{p-1}}}$

$1 = \frac{1}{p} + \frac{1}{q} = \frac{1}{p} + \frac{p-1}{p} \Rightarrow q = \frac{p}{p-1}$

$$= {}_x\int \left({}_y\int F^p d\nu\right)^{1/p} \cdot \left({}_y\int g^{p-1 \cdot p/p-1} d\nu\right)^{\frac{p-1}{p}} d\mu$$

$$= \left({}_y\int g^p d\nu\right)^{\frac{p-1}{p}} \cdot {}_x\int \left({}_y\int F^p d\nu\right)^{1/p} d\mu$$

pull g integral out, ind. of x (?) } ck: TIM

$$= \left({}_y\int \left({}_x\int F d\mu\right)^p d\nu\right)^{\frac{p-1}{p}} \cdot {}_x\int \left({}_y\int F^p d\nu\right)^{1/p} d\mu$$

$$\Rightarrow {}_y\int \left({}_x\int F^p d\mu\right) d\nu \leq \boxed{\left({}_y\int \left({}_x\int F d\mu\right)^p d\nu\right)^{\frac{p-1}{p}}} \cdot {}_x\int \left({}_y\int F^p d\nu\right)^{1/p} d\mu$$

divide both sides by this

$$\left({}_y\int \left({}_x\int F^p d\mu\right) d\nu\right)^{1-\frac{p-1}{p}} \leq {}_x\int \left({}_y\int F^p d\nu\right)^{1/p} d\mu$$

$$\left({}_y\int \left({}_x\int F^p d\mu\right) d\nu\right)^{1/p} \leq {}_x\int \left({}_y\int F^p d\nu\right)^{1/p} d\mu \quad \checkmark$$

4. Let $(X, \mathcal{B}(X), \mu)$ be a measure space. If $f_n, g_n, f, g \in L^1(X, \mu)$, and

(i) $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ a.e.;

(ii) $|f_n| \leq g_n$ for all $n \in \mathbb{N}$;

(iii) $\lim_{n\to\infty} \int_X g_n(x)d\mu(x) = \int_X g(x)d\mu(x)$. Then one has

$$\boxed{\lim_{n\to\infty} \int_X f_n(x)d\mu(x) = \int_X f(x)d\mu(x).}$$

Generalized DCT   claim

mimic proof of reg DCT in Folland

$|f_n| \leq g_n \ \forall n \Rightarrow \begin{cases} f_n \leq g_n \Rightarrow 0 \leq g_n - f_n \Rightarrow \{g_n - f_n\} \subset L^+ \\ -f_n \leq g_n \Rightarrow 0 \leq g_n + f_n \Rightarrow \{g_n + f_n\} \subset L^+ \end{cases}$

using Fatou's Lemma:

$\int g - f = \int \liminf (g_n - f_n) = \int g - \limsup f_n \leq \int g - \limsup \int f_n$

$\int g + f = \int \liminf (g_n + f_n) = \int g + \liminf f_n \leq \int g + \liminf \int f_n$

$\int g - \int f \leq \int g - \limsup \int f_n \Rightarrow \int f \geq \limsup \int f_n$

$\int g + \int f \leq \int g + \liminf \int f_n \Rightarrow \int f \leq \liminf \int f_n$

So $\limsup \int f_n \leq \int f \leq \liminf \int f_n$

thus $\lim \int f_n = \int f$ ✓

5. Assume that $E \subseteq \mathbb{R}$ is Lebesgue measurable and $0 < m(E) < \infty$.

(A) Show that if E is bounded and $m(E) = p > 0$, then for each $q \in (0, p)$ there is a measurable set $B \subseteq E$ of measure q.

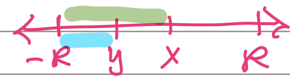
(B) Prove that for any $0 < \alpha < 1$ there is an open interval I such that

$$\alpha m(I) \leq m(E \cap I).$$

a.) $m(E) = p \in (0,\infty)$ and $E \subseteq \mathbb{R}$. Then $\exists R$ s.t. $E \subseteq [-R, R]$.
Let $E_x = [-R, x]$, and observe that $f(x) = m(E_x \cap E)$ is a continuous, nondecreasing function.

↳ Proof: Nondecreasing is trivial.
For continuity, WTS: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$.
Let $\varepsilon > 0$ and let $|x - y| < \delta = \varepsilon$. Then

$$\begin{aligned} |f(x) - f(y)| &= |m(E_x \cap E) - m(E_y \cap E)| \\ &= |m([-R, x] \cap E) - m([-R, y] \cap E)| \\ &= |m(\underbrace{[y, x] \cap E}_{})| < \varepsilon \checkmark \end{aligned}$$



$\subseteq [y, x]$, so by monotonicity of measure
$m([y,x] \cap E) \leq m([y,x]) = |x - y| < \varepsilon$

Since $f : [-R, R] \to [0, p]$ is continuous, by IVT for any $q \in (0, p)$ $\exists x$ s.t. $f(x) = q$, so $\exists E_x$ s.t. $m(E_x \cap E) = q$. Then $B = E_x \cap E$ satisfies the desired properties.

b.) AFSOC $\exists \alpha \in (0,1)$ s.t. $\alpha m(I) > m(E \cap I)$ for all open intervals I.
Let $\{I_i\}_{i=1}^{\infty}$ be a collection of disjoint intervals s.t.

- $E \subset \bigcup_{i=1}^{\infty} I_i$
- $m(\bigcup_{i=1}^{\infty} I_i) \leq m(E) + \varepsilon$

CR: TIM

Then observe that

$$\begin{aligned} m(E) &= m(E \cap (\textstyle\bigcup_{i=1}^{\infty} I_i)) \\ &= m(\textstyle\bigcup_{i=1}^{\infty} (E \cap I_i)) \\ &\leq \textstyle\sum_{i=1}^{\infty} m(E \cap I_i) \\ &< \textstyle\sum_{i=1}^{\infty} \alpha m(I_i) \\ &< \alpha (\textstyle\sum_{i=1}^{\infty} m(I_i)) \\ &= \alpha (m(\textstyle\bigcup_{i=1}^{\infty} I_i)) \\ &< \alpha (m(E) + \varepsilon) \end{aligned}$$

since I_i's are disjoint

$\Rightarrow m(E) < \alpha(m(E) + \varepsilon)$
Picking $\varepsilon < (\alpha - 1) m(E)$, we obtain:
$m(E) < \alpha m(E) + \alpha(\alpha - 1) m(E)$
$m(E) < \underbrace{\alpha^2}_{<1} m(E)$ ↯

We obtain a contradiction, so such an interval I must exist for any $\alpha \in (0,1)$.

FALL 2021

1. (a) For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, write down the definition of

$$\limsup_{n\to\infty} a_n.$$

(b) Prove that, for any sequence of Lebesgue measurable functions $f_n : \mathbb{R} \to [0,1]$, $n = 1, 2, \ldots$, the function

$$f = \limsup_{n\to\infty} f_n$$

is also Lebesgue measurable.

a.) $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k = \inf_n \sup_{k\geq n} a_k$

b.) measurable func: preimage of msable set is msable

To show $\limsup_{n\to\infty} a_n = \inf_n \sup_{k\geq n} a_k$ msable, must show that inf + sup are msable

i.) sup f_n msable

Lebesgue measure def on Borel sets. So <u>WTS</u>: preimage of Borel is Borel.

Let $E \subseteq \mathcal{B}([0,1])$

$\sup_n f_n^{-1}(E) = \sup_n (f_n^{-1}(E)) = \bigcup_{n=1}^{\infty} f_n^{-1}(E) \in \mathcal{B}([0,1])$ as a ctable union of msable sets in $[0,1]$ is in $\mathcal{B}([0,1])$

ctable as seq indexing is ctable

ii.) inf f_n msable

$\inf_n f_n^{-1}(E) = \inf_n (f_n^{-1}(E)) = \bigcap_{n=1}^{\infty} f_n^{-1}(E) \in \mathcal{B}([0,1])$ as a ctable intersection //

once again ctable

2. Consider the ℓ^1 space

$$\ell^1 = \{x = (x_1, x_2, \dots) : x_k \in \mathbb{R}, \|x\|_1 < \infty\}, \quad \|x\|_1 = \sum_{k=1}^{\infty} |x_k|,$$

and endow the space with the metric $\rho(x, y) = \|x - y\|_1$. Prove that

$$K = \{x = (x_1, x_2, \dots) : |x_k| \le k^{-2}, k = 1, 2, \dots\}$$

is a compact set in ℓ^1.

CR: TIM

compact = seq. cpt. in a metric space

To show that K is compact, WTS: every seq in K has a conv. subseq.

Let $\{x^n\} \subseteq K$. For all n & each k, $|x_k^n| \le k^{-2}$.

We note that $[-k^{-2}, k^{-2}]$ is compact in $\mathbb{R}^2$ for each k.

Then for each fixed k, $\{x_k^n\} \subseteq [-k^{-2}, k^{-2}]$, and as a seq in a compact set, $\exists$ conv. subseq $\{x_k^{n_j}\} \to x_k^* \in [-k^{-2}, k^{-2}]$.

seq in n

By Cantor's diag. argument, there exists a general subseq in n s.t. $\{x_k^n\} \to x_k^*$ for each k.

Then:

$$\begin{aligned}
\|x^*\|_1 = \|(x_1^*, x_2^*, \dots)\|_1 &= \textstyle\sum_{k=1}^{\infty} |x_k^*| \\
&= \textstyle\sum_{k=1}^{\infty} \lim |x_k^{n_j}| \\
&\le \liminf \textstyle\sum_{k=1}^{\infty} |x_k^{n_j}| \quad \text{by Fatou} \\
&\le \liminf \textstyle\sum_{k=1}^{\infty} k^{-2} \\
&< \infty
\end{aligned}$$

Then $x^* \in \ell^1$ & by construction $|x_k^*| \le k^{-2}$ $\forall k$. Then every seq. $\{x^n\} \subseteq K$ has a conv subseq in K. ✓

3. We say a function $g : [0,1] \to \mathbb{R}$ is Lipschitz if

$$\sup\left\{\frac{|g(x)-g(y)|}{|x-y|} : x, y \in [0,1], x \neq y\right\} < \infty.$$

Prove that, for any non-decreasing and absolutely continuous function $f : [0,1] \to \mathbb{R}$, there exists a sequence of functions $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, \ldots$, such that each f_n is non-decreasing, Lipschitz, and

$f_1 \le f_2 \le \ldots$ and $\lim_{n\to\infty} f_n = f$ everywhere on $[0,1]$.

monotone ↳ use MCT

FTC: abs cont on $[0,1]$ $\iff$

1.) $f(a) = f(0) + \int_0^a g(t)\,dt$ $\forall a \in [0,1]$, $g \in L^1$

2.) $f(a) = f(0) + \int_0^a f'(t)\,dt$

f diffable on $[0,1]$, $f' \in L^1$

(*) Since f nondec + abs cont, $f' \ge 0$

Hint: Define a helper seq. of simple funcs $\{\varphi_n\}$ s.t.

- $\varphi_n \le \varphi_{n+1}$ $\forall n$
- $\varphi_n \le f'$ $\forall n$
- $\varphi_n \to f'$ a.e.

Then let $f_n(x) = f(0) + \int_0^x \varphi_n(t)\,dt$ ← like $g \in L^1$

1.) Check Lipschitz

let $x, y \in [0,1]$. WLOG let $x < y$. ← not really needed

$$|f_n(x) - f_n(y)| = \left|\left(f(0) + \int_0^x \varphi_n(t)\,dt\right) - \left(f(0) + \int_0^y \varphi_n(t)\,dt\right)\right|$$
$$= \left|\int_x^y \varphi_n(t)\,dt\right|$$
$$\le |x-y| \cdot \|\varphi_n\|_{L^\infty} \quad \text{(M.L)}$$

$$\Rightarrow \frac{|f_n(x) - f_n(y)|}{|x-y|} \le \|\varphi_n\|_{L^\infty} < \infty \ \checkmark$$

↳ property of simple funcs.

2.) Check nondecreasing

Since each φ_n is nondec, $f_n(x) = f(0) + \int_0^x \varphi_n(t)\,dt$ is also nondecreasing as x increases ✓

3.) Check monotone

Since $\varphi_n \le \varphi_{n+1}$ $\forall n$ by construction, $f_n(x) = f(0) + \int_0^x \varphi_n(t)\,dt$ is also monotone

4.) Check convergence

WTS: $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.

$$\lim f_n(x) = \lim\left(f(0) + \int_0^x \varphi_n(t)\,dt\right)$$
$$= f(0) + \lim \int_0^x \varphi_n(t)\,dt$$
$$= f(0) + \int_0^x \lim \varphi_n(t)\,dt \quad \text{(by MCT since } \varphi_n \in L^1\ \forall n\text{) property of simple funcs.}$$
$$= f(0) + \int_0^x f'(t)\,dt \quad \text{(since } \varphi_n \to f' \text{ a.e.) by construction}$$
$$= f(x) \ \checkmark \quad \text{(by FTC)}$$

4. Let $f_n : [0,\infty) \to \mathbb{R}$, $n = 1, 2, \ldots$, be such that

$$\lim_{n\to\infty} f_n(x) = e^{-x} \text{ for every } x \in [0,\infty), \quad \text{conv. a.e.}$$

$$\sup_n \int_0^\infty |f_n(x)|^2 e^{-x}\,dx < \infty,$$

$$\limsup_{n\to\infty} \int_0^\infty |f_n(x)|\,dx \le 1.$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty |f_n(x) - e^{-x}|\,dx = 0. \quad \text{conv. in } L^1$$

limsup = - liminf f_n

since $f_n(x) \to e^{-x}$ a.e., $|f_n(x)| \to |e^{-x}|$ a.e.

using Fatou on $|f_n|$ ($\subset L^+$), we get

$\int_0^\infty \liminf |f_n| \le \liminf \int_0^\infty |f_n|$

$\int_0^\infty |e^{-x}| \le \liminf \int_0^\infty |f_n|$

$\int_0^\infty |e^{-x}| = 1$ (omitting bounds of integration moving forward for ease of notation)

Then $\int |e^{-x}| \le \liminf \int |f_n| \le \limsup \int |f_n| \le 1 = \int |e^{-x}|$

$\Rightarrow \int |e^{-x}| = \lim \int |f_n|$ $(=1)$

$\lim \int |f_n - e^{-x}| \le \lim \int |f_n| + |e^{-x}| = 1 + 1 = 2$ ῾̈

WTS: $\lim \int |f_n - e^{-x}| = 0$

can't use tri. ineq., gotta be more clever

Ideally $\lim \int |f_n - e^{-x}| = \lim \int |f_n| - \int e^{-x} = 1 - 1 = 0$

Try showing $\lim \int |f_n - e^{-x}| - |f_n| = -\int e^{-x} = -1$?

NEW

WTS: $\lim \int |f_n - e^{-x}| - |f_n| = -1$

would love to pull limit inside:

$\lim \int |f_n - e^{-x}| - |f_n| = \int \lim |f_n - e^{-x}| - |f_n| = \int -e^{-x} = -1$ ✓

Try using DCT! rev. tri. ineq.

$\big||f_n - e^{-x}| - |f_n|\big| \le |f_n - e^{-x} - f_n| = |-e^{-x}| = e^{-x} \in L^1$ (yay!)

and $\{|f_n - e^{-x}| - |f_n|\} \subseteq L^1$ since $\{f_n\} \subseteq L^1$ by this property

lastly $|f_n(x) - e^{-x}| - |f_n(x)| \to -e^{-x}$ a.e., so we may use DCT and achieve the desired result!

5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to [0,1]$ be $\mathcal{A}$-measurable.

(a) Let $\mathcal{B}_{[0,1]}$ denote the Borel σ algebra on $[0,1]$. Prove that the set

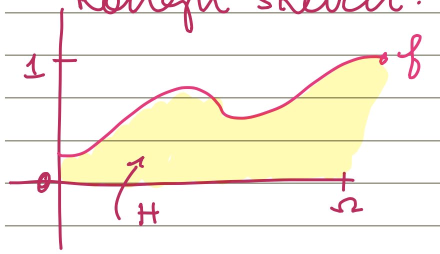
$$H = \{(x,y) : x \in \Omega, y \in [0, f(x)]\} \subset \Omega \times [0,1]$$

is $(\mathcal{A} \times \mathcal{B}_{[0,1]})$-measurable.

(b) Assume further that $\mu(\Omega) < \infty$, and let m denote the Lebesgue measure on $[0,1]$. Prove that

$$(\mu \times m)(H) = \int_\Omega f \, d\mu.$$

Rough sketch:



(over simplifying Ω as an interval)

a.) H is measurable if for each fixed $x \in \Omega$, $y \in [0,1]$, $H_x \in \mathcal{B}([0,1])$ & $H_y \in \mathcal{A}$
(i.e. each cross section is in the proper σ-alg.)

$H_x = \{ y \mid y \in [0, f(x)] \} \subseteq [0,1]$

$H_x = [0, f(x)]$, of the form $[0,b]$, which generates $\mathcal{B}([0,1])$, so

$H_x = [0, f(x)] \subseteq \mathcal{B}([0,1])$ ✓

$H_y = \{ x \mid f(x) \geq y \} \subseteq \Omega$

$H_y = f^{-1}([y,1])$. Since $[y,1] \subseteq [0,1]$ & is of form $[a,1]$, $[y,1] \subseteq \mathcal{B}([0,1])$.
Since f is measurable, the preimage of a measurable set is also measurable. Then H_y is $\mathcal{A}$-measurable ✓

b.) $\mu(\Omega) < \infty$, and H is $(\mathcal{A} \times \mathcal{B}[0,1])$-msable.

$(\mu \times m)(H) = \iint \chi_H \, d\mu \, dm$

$\in L^1(\Omega \times [0,1])$ since H is $(\mathcal{A} \times \mathcal{B}_{[0,1]})$-msable $\Rightarrow$ use Fubini!

$$(\mu \times m)(H) = \iint \chi_H \, d\mu \, dm$$

$$= \int_\Omega \int_0^1 \chi_{H^x} \, dm(y) \, d\mu(x)$$

$$= \int_\Omega \int_0^{f(x)} 1 \, dm(y) \, d\mu(x)$$

$$= \int_\Omega f(x) \, d\mu(x) \checkmark$$

SPRING 2022

1. Let A be a Cantor-like set defined as follows: Given $\alpha \in (0,1)$, we remove from $A_0 = [0,1]$ an open interval $(\frac{1}{2} - \frac{\alpha}{4}, \frac{1}{2} + \frac{\alpha}{4})$ and denote by A_1 the union of the two remaining closed intervals. Next we remove the open middle intervals of length $\frac{\alpha}{2^3}$ of the two intervals constituting A_1 and denote by A_2 the union of the four remaining closed intervals. We repeat the process with each of the four intervals, removing the open middle intervals of length $\frac{\alpha}{2^5}$. Continuing the process, we obtain the sequence $(A_n)_{n\geq 0}$ of sets, where A_n is the union of 2^n closed intervals, and we put $A = \bigcap_{n\geq 0} A_n$.

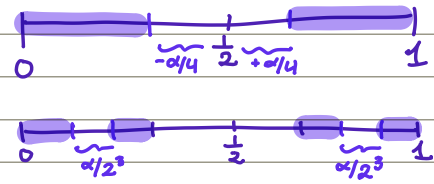
   (a) Show that $A \neq \varnothing$ and A is compact.

   (b) Show that A is nowhere dense, i.e. $\text{int}(\text{cl}(A)) = \varnothing$.

   (c) Show that A is perfect.

   (d) Show that A is uncountable.

   (e) Find the Lebesgue measure μ of A.



etc.

a.) $A \neq \emptyset$ as $0, 1 \in A$ (will never be part of a middle third, so never removed!)
$A \subset \mathbb{R}$, so use Heine-Borel to show cpt by showing A is closed + bounded. Obviously A is bounded since $A \subset [0,1]$, a bdd interval. Each $A_i = A_{i-1} \setminus U_i$ where U_i is an open interval, so each A_i is closed as the complement of open set(s). Since $A = \bigcap_{i=1} A_i$, so A is the intersection of closed sets, A is closed.

b.) A nowhere dense $\Rightarrow \text{int}(cl(A)) = \emptyset \Rightarrow \nexists\, U^{\text{open}} \in \text{int}(cl(A))$.
AFSOC $\exists\, U \in \text{int}(cl(A))$. Then $U \subset A_i$ for some i. However since U is a connected open interval, for some j, the removal of the middle interval from A_{i+j-1} removes some point(s) in U, and hence $U \not\subset A_{i+j}$ so $U \not\subset \bigcap_{i=1} A_i = A$. So $U \not\subset A$ ↯.

c.) Perfect = closed set where every pt is a lim pt
Closed is done ✓ For any $a \in A$, define $\{a_n\} \to a$ like so:
In A_i, let $[x_i, y_i]$ be the interval containing a. Then let $\{a_n\}$ be $x_1, y_1, x_2, y_2, \dots$
until, for some j, $x_j = y_j = a$. Then $\forall n \geq j$, $a_n = a$. Thus $\{a_n\} \to a$, and such a seq exists $\forall a \in A$
↳ $\{a_n\} = x_1, y_1, x_2, y_2, \dots, x_{j-1}, y_{j-1}, a, a, \dots$

d.) Note: $|C| = |\mathbb{R}| = 2^{\aleph}$ (uncountable)
At each iteration to construct A_i, each interval in A_{i-1} is split into two smaller intervals when the middle third is removed. At iteration i, there are 2^i intervals (plus the point at $\frac{1}{2}$ for $i \geq 1$). Then, in A, there are $2^{\aleph}$ intervals, so the intersection of the intervals (which forms A) is uncountable.

e.) $\mu([0,1]) = 1 = \mu(A) + \mu(A^c) \Rightarrow \mu(A) = 1 - \mu(A^c)$
$A^c = \bigcup_{i=1} B_{i,j}$ where $B_{i,j}$ is the j^{th} interval removed in iteration i to make A_{i+1}
$\mu(B_{i,j}) = \alpha / 2^{2i+1}$

So
A_1: cut $\frac{\alpha}{2}$
A_2: cut $\frac{\alpha}{2^3}$
A_3: cut $\frac{\alpha}{2^5}$

Each removed interval is disjoint from others, so
$\mu(\bigcup_{i=1} B_{i,j}) = \sum_{i=1}^{\infty} \mu(B_{i,j}) = \sum_{i=1}^{\infty} \alpha / 2^{2i+1} = \frac{\alpha}{2} \underbrace{\sum_{i=1}^{\infty} \frac{1}{2^{2i}}}_{\text{converges to } \frac{1}{3}} = \frac{\alpha}{2} \cdot \frac{1}{3}$

Then $\mu(A) = 1 - \frac{\alpha}{6}$

2. Let m denote the Lebesgue measure on $\mathbb{R}$. Suppose that f is nonnegative and measurable function on a set $A \subseteq \mathbb{R}$ such that $m(A) < \infty$. Prove that $f \in L^1(A)$ if and only if

$$\sum_{k=0}^{\infty} k\, m(A_k) < \infty,$$

where $A_k = \{x \in A : k \leq f(x) < k+1\}$.

≥ 0 as f nonneg.

* If f nonneg. + integrable, $\sum \int f = \int \sum f$
↳ like special case of F-T

($\Rightarrow$) assume $f \in L^1(A)$, then $\int_A |f| < \infty$.

$A = \bigcup_{k=0}^{\infty} A_k$ and each A_k disjoint by def. Then $\int_A = \int \sum_{k=0}^{\infty} \chi_{A_k} = \sum_{k=0}^{\infty} \int_{A_k}$

Then $\int_A |f| = \int_{A_1} |f| + \int_{A_2} |f| + \ldots = \sum_{k=0}^{\infty} \int_{A_k} |f|$

on A_k, $|f(x)| \underset{\uparrow}{=} f(x) \geq k$, so: $\sum_{k=0}^{\infty} \int_{A_k} |f| \geq \sum_{k=0}^{\infty} \int_{A_k} k = \sum_{k=0}^{\infty} k \cdot m(A_k)$

f nonneg.

Then $\infty > \int_A |f| \geq \sum_{k=0}^{\infty} k \cdot m(A_k) \Rightarrow \sum_{k=0}^{\infty} k \cdot m(A_k) < \infty$ ✓

($\Leftarrow$) assume $\sum_{k=0}^{\infty} k \cdot m(A_k) < \infty$.

Note $\int_A |f| \underset{\uparrow}{=} \sum_{k=0}^{\infty} \int_{A_k} |f|$. also note that $|f(x)| \overset{\text{nonneg.}}{=} f(x) < k+1$ on A_k, so

by same argument as above, $A = \bigcup_{k=0}^{\infty} A_k$ w/ A_k's disjt.

$$\sum_{k=0}^{\infty} \int_{A_k} |f| < \sum_{k=0}^{\infty} \int_{A_k} (k+1) = \sum_{k=0}^{\infty} (k+1)\, m(A_k) = \underbrace{\sum_{k=0}^{\infty} k \cdot m(A_k)}_{<\infty \text{ by assumption}} + \overbrace{\sum_{k=0}^{\infty} m(A_k)}^{<\infty}$$

$\Rightarrow \int_A |f| < \infty$ ✓

disjt. $\sum_{k=0}^{\infty} m(A_k)$
↳ $= m(\bigcup_{k=0}^{\infty} A_k)$
$= m(A) \underset{(\text{given})}{< \infty}$

3. Let m denote the Lebesgue measure on $\mathbb{R}$. Let $I = [0,1]$. Decide whether the sets A and B below are closed in $L^1(I, m)$, where

$$A := \left\{ f \in L^1(I,m) : \int_I |f(x)|^2 dm \geq 1 \right\}$$

and

$$B := \left\{ f \in L^1(I,m) : \int_I |f(x)|^2 dm \leq 1 \right\}$$

See Fall 2015 Prob #1

4. Let $(X, \mathcal{B}(X), \mu)$ be a finite measurable space. Assume that two sequences $\{f_n\}$ and $\{g_n\}$ of measurable functions satisfy $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ (converge measure). Show that the product $f_n g_n \xrightarrow{\mu} fg$ converge in the measure. Is the statement true is $\mu(X)$ is infinite?

conv in $\mu \Rightarrow \exists$ subseq. $\{f_{n_j}\} \to f$ a.e.
Is the reverse true? A: Only on finite m.s., else we can't use Egorov

Cl: If $\exists$ subseq. $\{f_{n_k}\} \to f$ a.e., then $f_n \xrightarrow{\mu} f$

Pf: AFSOC $f_n \not\xrightarrow{\mu} f$, so $\exists \varepsilon > 0$ & subseq. $\{f_{n_j}\}$ s.t. $\mu(\{x \mid |f_{n_j}(x) - f(x)| > \varepsilon\}) > \varepsilon$
↳ pick this ε + $\{f_{n_j}\}$

Egorov: If $\mu(X) < \infty$ + $\{f_{n_{j_k}}\} \to f$ a.e., then $\forall \varepsilon > 0$, $\exists E \subseteq X$ s.t. $\mu(E) < \varepsilon$ + $f_{n_{j_k}} \rightrightarrows f$ (uni.) on E^c

Since $\{f_{n_{j_k}}\} \to f$ a.e., $\exists E \subseteq X$ s.t. $f_{n_{j_k}} \rightrightarrows f$ on E^c
Then $\{x \mid |f_{n_{j_k}}(x) - f(x)| > \varepsilon\} \subseteq E$, and since $\mu(E) < \varepsilon$, by monotonicity of μ we have $\mu(\{x \mid |f_{n_{j_k}}(x) - f(x)| > \varepsilon\}) \leq \mu(E) < \varepsilon$

However, this contradicts the assumption that
$\mu(\{x \mid |f_{n_j}(x) - f(x)| > \varepsilon\}) > \varepsilon$
since $\{f_{n_{j_k}}\}$ is a subseq. of $\{f_{n_j}\}$
Hence, if $\mu(X) < \infty$, $f_n \xrightarrow{\mu} f \iff \exists \{f_{n_j}\}$ s.t. $f_{n_j} \to f$ a.e. $\square$

With this claim, we have the following:
$f_n \xrightarrow{\mu} f \Rightarrow \exists \{f_{n_j}\}$ s.t. $f_{n_j} \to f$ a.e.
$g_n \xrightarrow{\mu} g \Rightarrow \exists \{g_{n_k}\}$ s.t. $g_{n_k} \to g$ a.e.
Then for $\{f_n g_n\}$, $\exists$ subseq. $f_{n_\ell} g_{n_\ell}$ which conv a.e. to fg.
By our claim, then $f_n g_n \xrightarrow{\mu} fg$ ✓

This does not always work if $\mu(X) = \infty$.
Counterex: Let $X = \mathbb{R}$, $f_n(x) = \frac{1}{n}$, $g_n(x) = x$
$f_n \to 0$ in measure, $g_n \to x$ in measure
Then $\mu(\{\, |\underbrace{\tfrac{1}{n} \cdot x - 0}_{f_n g_n - fg}| > \varepsilon\}) = \infty$ $\forall n$, so
$f_n g_n \not\xrightarrow{\mu} fg$.

5. Let $\Lambda(\mathbb{R}^3)$ be the Lebesgue σ-algebra of $\mathbb{R}^3$, and let λ denote the three-dimensional Lebesgue measure on $\Lambda(\mathbb{R}^3)$. Let $\mathcal{P}(\mathbb{R}^3)$ denote the power set of $\mathbb{R}^3$. Clearly $\Lambda(\mathbb{R}^3) \subset \mathcal{P}(\mathbb{R}^3)$. Show that λ cannot be extended to $\mathcal{P}(\mathbb{R}^3)$.

Hint: You may use the Banach-Tarski paradox, which states the following:
Theorem *If U and V are arbitrary bounded open sets in $\mathbb{R}^n$, with $n \geqslant 3$, then there exists a $k \in \mathbb{N}$ and subsets $E_1, \dots, E_k, F_1, \dots, F_k$ of $\mathbb{R}^n$ such that*
(i) $E_j \cap E_l = \varnothing$ whenever $j \neq l$, and $\bigcup_j E_j = U$;
(ii) $F_j \cap F_l = \varnothing$ whenever $j \neq l$, and $\bigcup_j F_j = V$;
(iii) $E_j \sim F_j$ for all $j = 1, \dots, k$.
Here $\sim$ means Euclidean congruence, i.e. $A \sim B$ if A can be mapped into B by a combination of a translation, a rotation, and a reflection.

FALL 2022T

1. Let $f : [0,1] \to [0,1]$ be continuous and non-decreasing, with $f(0) = 0$ and $f(1) = 1$. For such an f, the derivative $f'(x)$ exists for Lebesgue almost every $x \in [0,1]$. Let dx denote Lebesgue measure.

a) Use Fatou's lemma to show that $\int_0^1 f'(x)dx \leq 1$.

b) Give an example of such a function f for which $\int_0^1 f'(x)dx < 1$, and explain why strict inequality holds.

a.) For Fatou: $\{f_n\} \subseteq L^+$ & $f_n \to f'$ so that

$$\int \liminf f_n \leq \liminf \int f_n$$
$$\int f'(x)\,dx \leq 1$$

also need $\liminf \int f_n \leq 1$

Recall $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

so try $f_n(x) = \frac{f(x+\frac{1}{n}) - f(x)}{1/n}$ so $\lim f_n = f'$ ✓

$$= n\left(f(x+\tfrac{1}{n}) - f(x)\right)$$

measure/integral will get too big, so maybe

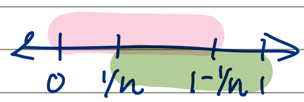
$$f_n(x) = n\left(f(x+\tfrac{1}{n}) - f(x)\right)\chi_{\cancel{[1/n,1]}\,[0,1-1/n]} \qquad \lim f_n = f'(x)\,\chi_{[0,1]}$$

$$\int f_n(x)\,dx = \int n\left(f(x+\tfrac{1}{n}) - f(x)\right)\chi_{\cancel{[1/n,1]}\,[0,1-\frac{1}{n}]}\,dx$$

$$= \int_{0}^{1-\frac{1}{n}} n\left(f(x+\tfrac{1}{n}) - f(x)\right)dx$$

$$= n \int_{0}^{1-\frac{1}{n}} f(x+\tfrac{1}{n})\,dx - \int_{0}^{1-\frac{1}{n}} f(x)\,dx$$

$$= n\left[\int_{\frac{1}{n}}^{1} f(x)\,dx - \int_0^{1-1/n} f(x)\,dx\right]$$



$$= n\left[\underbrace{\int_{1-1/n}^{1} f(x)\,dx}_{\leq \int_{1-1/n}^{1} f(1)\,dx} - \underbrace{\int_0^{1/n} f(x)\,dx}_{\geq \int_0^{1/n} f(0)\,dx}\right]$$

$$\Rightarrow \leq n\left(\int_{1-1/n}^{1} f(1)\,dx - \int_0^{1/n} f(0)\,dx\right)$$

$$\leq n\left(f(1)\left(1-\left(1-\tfrac{1}{n}\right)\right) - f(0)\left(\tfrac{1}{n} - 0\right)\right)$$

$$\leq n\left(1\left(\tfrac{1}{n}\right) - 0\left(\tfrac{1}{n}\right)\right)$$

$$\leq n \cdot \tfrac{1}{n} = 1 \checkmark \qquad \Rightarrow \liminf \int f_n \leq 1$$

so $\int f'(x)\,dx = \int \liminf f_n \leq \liminf \int f_n \leq 1$ ✓

b.) let $f(x)$ be the Cantor function. Then $f(x)$ is continuous & non-decreasing w/ $f(0)=0$ & $f(1)=1$, but $f(x)$ is constant Lebesgue a.e. then $f'(x) = 0$ Lebesgue-a.e., so $\int_0^1 f'(x)\,dx = 0 < 1$

2. Let dt denote Lebesgue measure on $\mathbb{R}$, and suppose $f \geqslant 0$ is Lebesgue integrable over $\mathbb{R}_+$, i.e.

$$\int_0^\infty f(t)dt < \infty.$$

Assume that $h > 0$.
Prove that

$$\lim_{h\to 0^+} \int_h^\infty \left(\frac{1}{h}\int_{x-h}^x f(y)dy\right)dx = \int_0^\infty f(t)dt.$$

Hint: 2 integrals tells me Fubini-Tonelli time!
Need $f \in L^+(X\times Y)$ or $f \in L^1(X\times Y)$

$$\int_h^\infty \frac{1}{h}\int_{x-h}^x f(y)dy = \int_0^\infty \frac{1}{h}\int_0^\infty f(y)\,\chi_{[x-h,x]}\,dy\,\chi_{[h,\infty)}\,dx$$

now we are in $L^+(X\times Y)$

$$= \frac{1}{h}\int_0^\infty\int_0^\infty f(y)\,\chi_{[x-h,x]}(y)\,dy\,\chi_{[h,\infty)}(x)\,dx$$

By Tonelli:

$$= \frac{1}{h}\int_0^\infty\int_0^\infty f(y)\,\chi_{[h,\infty)}(x)\,dx\,\chi_{[x-h,x]}(y)\,dy$$

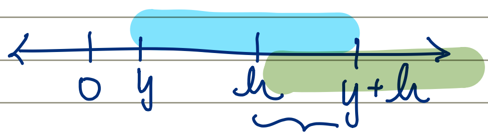
$$\left.\begin{array}{l} x-h \le y \le x \\ x \le y+h \le x+h \end{array}\right\} \Rightarrow y \le x \le y+h$$

$$= \frac{1}{h}\int_0^\infty f(y)\left(\int_0^\infty \chi_{[h,\infty)\cap[y,y+h]}(x)\,dx\right)dy$$

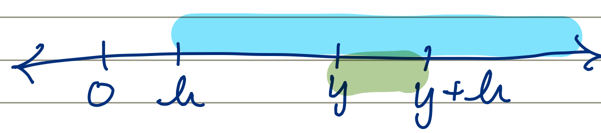
$$= \frac{1}{h}\int_0^\infty f(y)\, m([h,\infty)\cap[y,y+h])\,dy$$

$$= \frac{1}{h}\left[\int_0^h f(y)\,m([h,\infty)\cap[y,y+h])\,dy + \int_h^\infty f(y)\,m([h,\infty)\cap[y,y+h])\,dy\right]$$

$\hookrightarrow y \in [0,h]$ so $\qquad\qquad\qquad\qquad \hookrightarrow y \in [h,\infty)$ so



len = y



len = h

$$= \frac{1}{h}\left(\int_0^h f(y)\cdot y\,dy + \int_h^\infty f(y)\cdot h\,dy\right)$$

$$= \underbrace{\frac{1}{h}\int_0^h f(y)\,y\,dy}_{A} + \underbrace{\int_h^\infty f(y)\,dy}_{B}$$

$$\lim_{h\to 0^+}(B) = \lim_{h\to 0^+}\int_0^\infty \chi_{[h,\infty)}f(y)\,dy = \lim_{n\to\infty}\int_0^\infty \underbrace{\chi_{[\frac{1}{n},\infty)}f(y)}_{\in L^+}\,dy$$

by MCT

$$= \int_0^\infty \lim_{n\to\infty}\chi_{[\frac{1}{n},\infty)}f(y)\,dy = \int_0^\infty \chi_{[0,\infty)}f(y)\,dy = \int_0^\infty f(y)\,dy \checkmark$$

For (A), $y \in [0,h]$ so $y/h \le 1$, Then

$$\frac{1}{h}\int_0^h f(y)\,y\,dy \le \int_0^h f(y)\,dy = \int_0^\infty \underbrace{f(y)\,\chi_{[0,h]}\,dy}_{\to 0 \text{ as } h \to 0^+}$$

Then

$$\lim_{h\to 0^+}\int_0^\infty \frac{1}{h}\int_x^{x-h} f(y)\,dy = \int_0^\infty f(y)\,dy \quad ✓$$

3. Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence satisfying $\lim_{n\to\infty} na_n = 0$. Define

$$S_n := \sum_{k=1}^{n} a_k$$

and

$$\sigma_n := \frac{1}{n}\sum_{k=1}^{n} S_k.$$

Assume that

$$\lim_{n\to\infty} \sigma_n = A.$$

Prove that

$$\lim_{n\to\infty} S_n = A.$$

Given: $\lim_{n\to\infty} \sigma_n = A$

WTS: $\lim_{n\to\infty} S_n = A$

Hint: Show $S_n \to A$ by showing $|S_n - A| \to 0$ by actually showing $|S_n - \sigma_n| \to 0$

$$\begin{aligned}
|S_n - \sigma_n| &= \left|\sum_{k=1}^n a_k - \frac{1}{n}\sum_{k=1}^n S_k\right| \\
&= \left|\sum_{k=1}^n a_k - \frac{1}{n}\sum_{k=1}^n \sum_{i=1}^k a_i\right| \\
&= \left|\sum_{k=1}^n a_k - \frac{1}{n}\sum_{k=1}^n (n-k+1)a_k\right| \\
&= \left|\sum_{k=1}^n a_k - \sum_{k=1}^n \frac{n-k+1}{n} a_k\right| \\
&= \left|\sum_{k=1}^n a_k - \sum_{k=1}^n a_k + \frac{-k+1}{n} a_k\right| \\
&= \left|-\sum_{k=1}^n \frac{-k+1}{n} a_k\right| \\
&= \left|\sum_{k=1}^n \frac{k-1}{n} a_k\right|
\end{aligned}$$

$\sum_{k=1}^n \sum_{i=1}^k a_i$ → $n-(k-1)$ a_k's

$k=1$: a_1 — n a_1's

$k=2$: $a_1 + a_2$ — $n-1$ a_2's

$k=3$: $a_1 + a_2 + a_3$ — $n-2$ a_3's

$\vdots$ — $\vdots$

$k=n$: $a_1 + \dots + a_n$ — 1 a_k

$= \sum_{k=1}^n (n-k+1)a_k$

Since $\lim_{n\to\infty} na_n = 0$, $\exists N$ s.t. $\forall n \ge N$, $|na_n| < \varepsilon$.

Separate sum:

$$\left|\sum_{k=1}^n \frac{k-1}{n} a_k\right| = \left|\underbrace{\sum_{k=1}^N \frac{k-1}{n} a_k}_{\substack{\text{Finite sum, so} \\ \to 0 \text{ as } n\to\infty}} + \sum_{k=N+1}^n \frac{k-1}{n} a_k\right|$$

$$\begin{aligned}
\lim_{n\to\infty}|S_n - \sigma_n| &= \lim_{n\to\infty}\left|\sum_{k=1}^n \frac{k-1}{n} a_k\right| = \lim_{n\to\infty}\left|\sum_{k=1}^N \frac{k-1}{n} a_k + \sum_{k=N+1}^n \frac{k-1}{n} a_k\right| \\
&\le \underbrace{\lim_{n\to\infty}\left|\sum_{k=1}^N \frac{k-1}{n} a_k\right|}_{=0} + \lim_{n\to\infty}\left|\sum_{k=N+1}^n \frac{k-1}{n} a_k\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{n}\sum_{k=N+1}^n \underbrace{\frac{k-1}{k}}_{\substack{\approx 1 \\ \text{as } k\to\infty}} \cdot \underbrace{ka_k}_{\to 0 \text{ as } k\to\infty}\right| = 0 \quad ✓
\end{aligned}$$

→ has ctable dense subset

4. Consider a separable real Hilbert space H. (For the sake of concreteness, you may let dx be Lebesgue measure on the real line, and consider $H := L^2([0,1], dx)$, the equivalence classes of real Lebesgue square integrable functions on $[0,1]$ that differ at most on a null set, with inner product $\langle f, g\rangle := \int_{[0,1]} f(x)g(x)dx$ and norm $\|f\| := \left(\int_{[0,1]} |f(x)|^2 dx\right)^{\frac{1}{2}}$.)

a) Let $\{f_n\}_{n\in\mathbb{N}} \subset H$ be an orthonormal sequence. Show that $\lim_{n\to\infty}\langle g, f_n\rangle = 0$ for every $g \in H$. } ortho seq conv. weakly to 0

b) Now let $\{f_n\}_{n\in\mathbb{N}} \subset H$ be a sequence and $f \in H$ such that for every $g \in H$, one has $\lim_{n\to\infty}\langle g, (f_n - f)\rangle = 0$. Suppose that $\|f_n\| \to \|f\| \geqslant 0$ as $n \to \infty$. Show that $\|f_n - f\| \to 0$ as $n \to \infty$.

↳ conv weakly

upgrade weak conv to strong conv w/ norms

$\|\sum_1^n x_j\|^2 = \sum_1^n \|x_j\|^2$ in H when $x_j \perp x_k \;\; \forall j \neq k$

$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\langle x,y\rangle$

$\|x+y\|^2 \leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$

$\|x\| = \sqrt{\langle x,x\rangle}$ $\qquad |\langle x,y\rangle| \leq \|x\|\cdot\|y\|$

a.) orthonormal seq: $\|f_n\| = 1 \;\forall n$ and $f_n \perp f_m$ whenever $n \neq m$ (i.e. $\langle f_n, f_m\rangle = 0 \;\forall n \neq m$)

Bessel's Inequality: $\{u_\alpha\}$ orthonormal set in H, then for any $x \in H$: $\sum_\alpha |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2$ and $\langle x_\alpha, u\rangle \neq 0$ is countable

$\Rightarrow \forall g \in H: \sum_{n=1}^\infty |\langle g, f_n\rangle|^2 \leq \|g\|^2$

Pf: $0 \leq \|g - \sum_{n=1}^\infty \langle g, f_n\rangle f_n\|^2$ by def of norm

$= \|g\|^2 - 2\langle g, \sum_{n=1}^\infty \langle g, f_n\rangle f_n\rangle + \|\sum_{n=1}^\infty \langle g, f_n\rangle f_n\|^2$

$= \|g\|^2 - 2\sum_{n=1}^\infty |\langle g, f_n\rangle|^2 + \sum_{n=1}^\infty |\langle g, f_n\rangle|^2$

$= \|g\|^2 - \sum_{n=1}^\infty |\langle g, f_n\rangle|^2$

Since $g \in \underbrace{H}_{=L^2([0,1])}$, $\|g\|^2 < \infty$, so $\sum_{n=1}^\infty |\langle g, f_n\rangle|^2 < \infty$
sum converges, so $|\langle g, f_n\rangle| \to 0$ as $n \to \infty$
$\Rightarrow \langle g, f_n\rangle \to 0$ ✓

b.) $\|f_n - f\|^2 = \langle f_n - f, f_n - f\rangle = \|f_n\|^2 + \|f\|^2 - 2\,\mathrm{Re}\langle f_n, f\rangle$
as $n \to \infty$, $\|f_n\| \to \|f\|$ so $\|f_n\|^2 \to \|f\|^2$
as $n \to \infty$, $\langle g, f_n - f\rangle \to 0 \;\; \forall g \in H$
$\Rightarrow \langle g, f_n\rangle - \langle g, f\rangle \to 0$
$\Rightarrow \langle g, f_n\rangle \to \langle g, f\rangle$
let $g = f$: $\langle f, f_n\rangle \to \langle f, f\rangle = \|f\|^2$
$\|f_n - f\|^2 \to \|f\|^2 + \|f\|^2 - 2\|f\|^2 = 0$ ✓

5. Prove that if f is Lebesgue integrable on A, then, given $\varepsilon > 0$, there is $\delta > 0$ such that $\int_B |f(x)|dx < \varepsilon$ whenever $B \subseteq A$ and $|B| < \delta$.

Hint: approx w/ simple funcs
$f = \sum_{i=1}^n a_i \chi_{E_i}$ w/ E_i's disjt & $\bigcup_{i=1}^n E_i = A$

Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{\sum_{i=1}^n |a_i|}$, then when $m(B) < \delta$:

$$\begin{aligned}
{}_B\!\int |f| &= {}_B\!\int \left| \sum_{i=1}^n a_i \chi_{E_i} \right| dx \\
&\leq {}_B\!\int \sum_{i=1}^n |a_i \chi_{E_i}| dx \\
&= \sum_{i=1}^n {}_B\!\int |a_i \chi_{E_i}| dx \\
&= \sum_{i=1}^n {}_A\!\int |a_i| \chi_{E_i \cap B} dx \\
&= \sum_{i=1}^n |a_i| \cdot m(E_i \cap B)
\end{aligned}$$

} nonneg integrand, so swap $\sum$ & $\int$ (special case of F-T)

$E_i \cap B \subseteq B$, so by monotonicity $m(E_i \cap B) \leq m(B) < \delta$
$\Rightarrow m(E_i \cap B) < \delta \quad \forall i$

$$\begin{aligned}
\Rightarrow {}_B\!\int |f| &< \sum_{i=1}^n |a_i| \cdot \delta \\
&= \sum_{i=1}^n |a_i| \cdot \frac{\varepsilon}{\sum_{i=1}^n |a_i|} \\
&= \varepsilon
\end{aligned}$$

$\Rightarrow {}_B\!\int |f| < \varepsilon$

Suppose $f \in L^1(A)$. By density of simple funcs in L^1, $\exists$ a simple func g s.t. $\|f-g\|_{L^1} < \varepsilon$. Then find the appropriate $\delta > 0$ as specified above s.t. whenever $m(B) < \delta$, ${}_B\!\int |g| < \varepsilon$. Then:

$${}_B\!\int |f| \leq {}_B\!\int |f-g| + |g| < \varepsilon + \varepsilon = 2\varepsilon \checkmark$$

SPRING 2023

mark says this exam is weird, take w/ a grain of salt

1. Let f be a function in $C^2(\mathbb{R}^2)$. Assume that f is periodic:

$$f(x+1,y) = f(x,y+1) = f(x,y) \quad \forall\, (x,y) \in \mathbb{R}^2.$$

Prove that

$$\int_0^1 \int_0^1 \det(D^2 f)\,dx\,dy = 0,$$

where $D^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$.

(**Hint:** Do it first under a stronger assumption that $f \in C^3(\mathbb{R}^2)$.)

then we can integrate by parts!

→ This is an adv. calc, PDE type question.
↳ Probably will never come up again

First suppose $f \in C^3(\mathbb{R}^2)$. assume f is periodic

$$\int_0^1\int_0^1 \det(D^2 f)\,dx\,dy \overset{\text{int by parts}}{=} \int_0^1\int_0^1 \underbrace{\frac{\partial^2 f}{\partial x^2}}_{dv} \underbrace{\frac{\partial^2 f}{\partial y^2}}_{u} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 dx\,dy$$

$$= -\int \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 dx\,dy - \int_0^1\int_0^1 \frac{\partial f}{\partial x}\frac{\partial^3 f}{\partial x y^2}\,dx\,dy + \underbrace{\int_0^1 \left(\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x}(1,y) - \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x}(0,y)\right) dy}_{0 \text{ by periodicity}}$$

$$= -\int_0^1\int_0^1 \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 dx\,dy - \int_0^1\int_0^1 \underbrace{\frac{\partial f}{\partial x}}_{u} \underbrace{\frac{\partial^3 f}{\partial x y^2}}_{dv}\,dy\,dx \quad \text{Fubini Tonelli}$$

int by parts again

$$= -\int_0^1\int_0^1 \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 dx\,dy + \int_0^1\int_0^1 \frac{\partial^2 f}{\partial x y}\frac{\partial^2 f}{\partial x y}\,dy\,dx$$

etc.

Then need to generalize by approximating $C^2(\mathbb{R}^3)$ periodic funcs by $C^3(\mathbb{R}^2)$ periodic funcs
use convolutions or mollifying. DEFINITELY not covered in 501.
Talk to a PDE person if you're interested in more

2. Let $N \in \mathbb{N}$, the set of positive integers, and let $\{f_n\} \subset L^1(\mathbb{R}^N)$ be a sequence of functions satisfying

(a) $f_n \to f$ in $L^1(\mathbb{R}^N)$,

(b) $\{f_n\}$ is uniformly equicontinuous in $\mathbb{R}^N$, that is, for every $\epsilon > 0$ there exists a constant $\delta = \delta(\epsilon) > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \geq 1$, $x, y \in \mathbb{R}^N$, and $|x - y| < \delta$.

Prove that

$$\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |f_n(x)| = 0.$$

arzela-ascoli: $\{f_n\}$ equicont. & ptwise bdd
then $\exists \{f_{n_k}\} \to f$ on cpt sets

3. Let $f \in L^1([0,1])$, and let $\{f_n\}$ be a sequence of functions converging to f almost everywhere in $[0,1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 \Big| |f_n - f| - |f_n| + |f| \Big| dx = 0.$$

assume $\{f_n\}$ msable

Note: $||a|-|b|| \leq |a-b|$
$||a+b|-|b|| \leq |a|$

Hint from Mark: This is a trick w/ rev tri ineq.

$$||f_n - f| - |f_n| + |f|| \underset{(\text{reg } \Delta)}{\leq} ||f_n - f| - |f_n|| + \underbrace{||f||}_{=|f|} \underset{(\text{rev. } \Delta)}{\leq} |f_n - f - f_n| + |f| = \underbrace{2|f|}_{\in L^1}$$

Let $g_n = ||f_n - f| - |f_n| + |f||$

Note that each g_n is dominated by $2|f|$, regardless of n.

Then, since g_n msable (made up of msable pieces) & dominated by an L^1 func, $\{g_n\} \subseteq L^1$. Then, by DCT:

$\lim \int g_n = \int \lim g_n$

$\lim ||f_n - f| - |f_n| + |f||$

$\leq \underbrace{\lim ||f_n - f||}_{\to 0} + \underbrace{\lim |-|f_n| + |f||}_{\to 0\ (?)}$ so $\to 0$ ✓

$\mu([0,1]) = 1 < \infty$

4. Let $\{f_n(x)\}$ and $f(x)$ be continuous functions on $[0,1]$, and let g be a continuous function on $(-\infty,\infty)$. Assume that $\{f_n(x)\}$ converges to $f(x)$ in $[0,1]$ in Lebesgue measure. Prove that $\{g(f_n(x))\}$ converges to $g(f(x))$ in $[0,1]$ in Lebesgue measure.

$\{f_n(x)\} \to f(x)$ in measure
$\Rightarrow \exists$ subseq $\{f_{n_j}\} \to f$ a.e.

Seq conv iff for any subseq, $\exists$ further subseq which conv to x

Mark Hint: Egorov!
Since $X = [0,1]$ + $\mu(X) = 1 < \infty$ and $\{f_{n_j}\} \to f$ a.e., for any $\varepsilon' > 0$, $\exists E \subseq X$ s.t. $\mu(E) < \varepsilon'$
+ $\underbrace{f_{n_j} \rightrightarrows f}$ on E^c

$\forall \varepsilon > 0, \exists N$ s.t. $\forall n_j \geq N$, $|f_{n_j}(x) - f(x)| < \varepsilon \quad \forall x$

To show $g(f_n(x)) \to g(f(x))$ in μ, WTS: $\mu(\{x \mid |g(f_n(x)) - g(f(x))| \geq \delta\}) < \varepsilon$
g is continuous, so $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|a-b| < \delta$, $|g(a) - g(b)| < \varepsilon$

Take a further subseq. $\{f_{n_{j_k}}\}$ of $\{f_{n_j}\}$, + note since $f_{n_j} \to f$ a.e., $f_{n_{j_k}} \to f$ a.e.
Let $\varepsilon > 0$, $a = f_{n_{j_k}}(x)$, $b = f(x)$. By continuity of g, identify δ s.t. whenever $|a-b| < \delta$, $|g(a) - g(b)| < \varepsilon$.
On E^c, by uni conv of f_{n_j} to f, identify N s.t. $\forall n_j \geq N$, $|f_{n_{j_k}}(x) - f(x)| = |a-b| < \delta$.
Then $\forall n_{j_k} \geq N$, $|f_{n_{j_k}}(x) - f(x)| < \delta \quad \forall x$, so $|g(f_{n_{j_k}}(x)) - g(f(x))| < \varepsilon \quad \forall x \in E^c$.
So $\{x \mid |g(f_{n_{j_k}}(x)) - g(f(x))| \geq \varepsilon\} \subseteq E^c$, and by monotonicity of Lebesgue measure:

$$\mu(\{x \mid |g(f_{n_{j_k}}(x)) - g(f(x))| \geq \varepsilon\}) \leq \mu(E^c) < \varepsilon'$$

True for any $\varepsilon' > 0$, so $\mu(\{x \mid |g(f_{n_{j_k}}(x)) - g(f(x))| \geq \varepsilon\}) \to 0$ as $n \to \infty$
so $g(f_n) \to g(f)$ in measure ✓

5. Let $\{g_n\} \in C^1((0,1))$, and $g \in C^2((0,1))$. Assume that

$$\lim_{n\to\infty} \sup_{0<x<1} (|g_n(x) - g(x)| + |g_n'(x) - g'(x)|) = 0.$$

Prove that, for any sequence of positive numbers $\{h_n\}$ going to 0, and for any $0 < a < b < 1$, it holds

$$\lim_{n\to\infty} \frac{1}{(h_n)^2} \int_a^b [g_n(x+h_n) + g_n(x-h_n) - 2g_n(x)]dx = \int_a^b g''(x)dx.$$

FALL 2023

1. Let f_n for $n \in \mathbb{N}$ and f be complex-valued measurable functions on a measure space (X, μ) and suppose that f_n converges to f in measure. Then some subsequence of f_n converges to f pointwise μ-a.e.

Folland 2.4, Thm 2.30

$f_n \to f$ conv. in measure $\qquad \mu(\{x \mid |f_n(x)-f(x)| \geq \varepsilon\}) < \varepsilon \quad \forall n \geq N$ for some $N \in \mathbb{N}$

$\mu(\{x \mid |f_n(x)-f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$

Pick subseq. $\{f_{n_j}\}$ s.t.

$E_j = \{x \mid |f_{n_j}(x)-f(x)| \geq 2^{-j}\}$ $\qquad$ clearly $\mu(E_j) \leq 2^{-j} \longrightarrow 0$ as $j \to \infty$

Then $\sum_{j=1}^{\infty} \mu(E_j) = \underbrace{\sum_{j=1}^{\infty} 2^{-j}}_{\text{converges}} < \infty$

$\Rightarrow \mu(\{x \mid |f_{n_j}(x)-f(x)| \geq 2^{-j}\}) \longrightarrow 0$ as $j \to \infty$ $\qquad$ (fewer "further away" pts.)

$\Rightarrow |f_{n_j}(x) - f(x)| \to 0$ $\qquad$ (" then dist shrinks)

$\Rightarrow f_{n_j}(x) \to f(x)$ ptwise (a.e.) ✓

2. Let $a_n \geq 0$ for $n \in \mathbb{N}$, and let $0 < p < q$.

(i) Prove that

$$\left(\sum_{n=1}^{\infty} a_n^q\right)^{1/q} \leq \left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}.$$

(ii) Prove that for every $N \in \mathbb{N}$ we have

$$\left(\sum_{n=1}^{N} a_n^p\right)^{1/p} \leq N^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{n=1}^{N} a_n^q\right)^{1/q}.$$

$\frac{N^{1/p}}{N^{1/q}} = N^{q/p}$

i.) $(\sum_{n=1}^{\infty} a_n^q)^{1/q} \leq \sum_{n=1}^{\infty} (a_n^q)^{1/q} = \sum_{n=1}^{\infty} a_n$ $\qquad$ **CR: BENZI**

$(\sum_{n=1}^{\infty} a_n^q)^{1/q \cdot p/p} \overset{\text{tri ineq.}}{\leq} (\sum_{n=1}^{\infty} (a_n^q)^{p/q})^{1/q} = (\sum_{n=1}^{\infty} (a_n)^p)^{1/p}$ ✓

cleverly only once this part inside

ALT: (CR: ERIK)

6.11 Proposition. *If A is any set and $0 < p < q \leq \infty$, then $l^p(A) \subset l^q(A)$ and $\|f\|_q \leq \|f\|_p$.*

ii.) $\sum_{n=1}^{N} a_n \geq 0$ since each $a_n \geq 0$

6.12 Proposition. *If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p(\mu) \supset L^q(\mu)$ and $\|f\|_p \leq \|f\|_q \mu(X)^{(1/p)-(1/q)}$.*

$X = \{a_1, \ldots, a_n\}$

$\mu(X) < \infty$ (finite # pts.) $\qquad \mu(X) = N$

Given $0<p<q$, let $f(x) = x$ so $\int f(x)\,dx = \sum_{x \in X} x = \sum_{n=1}^{N} a_n$

$\|f\|_p^p = \int |f|^p = \int |f^p \cdot 1| = \|f^p \cdot 1\|_{L^1} \underset{\text{Hölder}}{\leq} \|f^p\|_{q/p} \cdot \|1\|_{\frac{p-q}{p}}$

$= (\int (f^p)^{q/p})^{p/q} = (\int f^q)^{p/q} \cdot N^{\frac{p}{p-q}}$

$\frac{q}{p} + \frac{p-q}{p} = 1$

$\|1\|_{\frac{p-q}{p}} = (\int_X 1^{\frac{p-q}{p}}\,dx)^{\frac{p}{p-q}} = (\int_X 1\,dx)^{\frac{p}{p-q}} = N^{\frac{p}{p-q}}$

$\int_X 1\,dx = \mu(X) = N$

$\|f\|_p^p \leq \|f\|_q^p \cdot N^{\frac{p}{p-q}}$

$\Rightarrow \|f\|_p \leq \|f\|_q \cdot N^{\frac{1}{p-q}}$

$\Rightarrow (\sum_{n=1}^{N} a_n^p)^{1/p} \leq (\sum_{n=1}^{N} a_n^q)^{1/q} N^{\frac{1}{p}-\frac{1}{q}}$ ✓

3. Show that if $f(x,y) = ye^{-(1+x^2)y^2}$ for each $x, y \in \mathbb{R}$, then

$$\int_0^\infty \left(\int_0^\infty f(x,y)dx \right) dy = \int_0^\infty \left(\int_0^\infty f(x,y)dy \right) dx.$$

Use the preceding equality to prove the formula

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$

Need $(X, \mathcal{M}, \mu)$ + $(Y, \mathcal{N}, \nu)$ σ-finite m.s.
σ-finite = if $X = \bigcup_1^\infty E_i$ where $E_i \subset \mathcal{M}$ $\forall i$, then $\mu(E_i) < \infty$ $\forall i$
↳ Leb. measure on $\mathbb{R}$ is σ-finite! ✓

Tonelli: $f \in L^+(X \times Y) \Rightarrow \iint f(x,y)\, dx\, dy = \iint f(x,y)\, dy\, dx$

$f(x,y) = \underbrace{y}_{\geq 0} \underbrace{e^{-(1+x^2)y^2}}_{\geq 0} \Big\} \geq 0$ $\quad X = [0,\infty)$, $Y = [0,\infty]$

$f(x,y) = g(x,y)\, h(x,y)$ w/ $g(x,y) = y$, $h(x,y) = e^{-(1+x^2)y^2}$ } both Lebesgue msable, so product $f(x,y)$ is, too.

Then by Tonelli: $\int_0^\infty \int_0^\infty f(x,y)\, dx\, dy = \int_0^\infty \int_0^\infty f(x,y)\, dy\, dx$ ✓

INT Y FIRST, THEN X

$$\int_0^\infty ye^{-(1+x^2)y^2} dy = \frac{-1}{2(1+x^2)} e^{-(1+x^2)y^2} \Big|_{y=0}^{\infty}$$
$$= \frac{-1}{2(1+x^2)} (e^{-\infty} - e^0)$$
$$= \frac{-1}{2(1+x^2)} (0-1) = \frac{1}{2(1+x^2)}$$

$$\int_0^\infty \frac{1}{2(1+x^2)} dx = \frac{1}{2} \arctan x \Big|_{x=0}^{x=\infty}$$
$$= \frac{1}{2}(\tan^{-1}(\infty) - \tan^{-1}(0))$$
$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}$$

INT X FIRST, THEN Y

$$\int_0^\infty ye^{-(1+x^2)y^2} dx = \frac{\sqrt{\pi}}{2} e^{-y^2}$$

$$\int_0^\infty \frac{\sqrt{\pi}}{2} e^{-y^2} dy = \frac{\pi}{4}$$
$$\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$$
$$2\int_0^\infty e^{-y^2} dy = \sqrt{\pi}$$
$$\int_{-\infty}^\infty e^{-y^2} dy = \sqrt{\pi} \quad ✓$$

Folland Ch. 2
Exercise 32

4. Let (X,μ) be a finite measure space. For every pair of measurable functions $f,g: X \to \mathbb{C}$ let

$$d(f,g) = \int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,d\mu(x).$$

(i) Show that d is a metric on the space of equivalence classes of measurable functions which differ only on measure zero sets.

(ii) Show that a sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$ converges in measure to a measurable function f if and only if $\lim_{n\to\infty} d(f_n,f)=0$.

(iii) Can one drop the assumption that $\mu(X)<\infty$ in (i)?

i.) metric space:
- $d(f,f)=0$
- $d(f,g)=d(g,f)$
- $d(f,h)\le d(f,g)+d(g,h)$

$d(f,f) = \int_X \frac{|f(x)-f(x)|}{1+|f(x)-f(x)|}dx = \int_X \frac{0}{1+0}dx = \int_X 0\,dx = 0$ ✓

$d(g,f) = \int_X \frac{|g(x)-f(x)|}{1+|g(x)-f(x)|} = \int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} = d(f,g)$ ✓

$\hookrightarrow |g(x)-f(x)| = |f(x)-g(x)|$

$d(f,h)\le d(f,g)+d(g,h)$

$0 \le d(f,g)+d(g,h)-d(f,h)$

RHS $= \int_X \frac{|f-g|}{1+|f-g|} + \int_X \frac{|g-h|}{1+|g-h|} + \int_X \frac{|f-h|}{1+|f-h|}$

$= \int_X \frac{|f-g|(1+|g-h|)(1+|f-h|)+\dots}{(1+|f-g|)(1+|g-h|)(1+|f-h|)}$

$= \int_X \frac{|f-g|+|g-h|-|f-h|}{(\text{positive stuff})}$

$= |f-g| + |f-g||g-h| + \cancel{|f-g||f-h|} + |f-g||g-h||f-h|$
$+ |g-h| + |g-h||f-g| + \cancel{|g-h||f-h|} + \cancel{''}$
$- |f-h| - \cancel{|f-h||f-g|} - \cancel{|f-h||g-h|} - \cancel{''}$
$= |f-g| + |g-h| - |f-h| + (\text{positive stuff})$

By Δ ineq: $|f-h|\le|f-g|+|g-h|$, then numerator ≥ 0, so $d(f,g)+d(g,h)-d(f,h)\ge 0$
$\Rightarrow d(f,g)+d(g,h) = d(f,h)$ ✓

ii.) $(f_n) \xrightarrow[\text{in measure}]{} f$ iff $\lim(f_n,f)=0$

($\Rightarrow$) $\mu(\{x \mid |f_n(x)-f(x)|\ge\varepsilon\}) \to 0$ as $n\to\infty$

$E_n = \{x \mid |f_n(x)-f(x)|\ge\varepsilon\}$

$\lim(f_n,f) = \lim \int_X \frac{|f_n-f|}{1+|f_n-f|} \le \lim \int |f_n-f|\,d\mu(x)$

$= \lim \int_{E_n} |f_n-f|\,d\mu(x) + \lim \int_{E_n^c} |f_n-f|\,d\mu(x)$

(separation works b/c $\mu(X)<\infty$)

$I_1 = 0$ $\quad I_2 \le M\cdot L = \varepsilon\cdot\mu(E_n^c)$

Send $\varepsilon\to 0$ (true $\forall\varepsilon$), then $I_1+I_2=0$

$\Rightarrow \lim(f_n,f)\le 0$
$\lim(f_n,f)=0$ ✓

($\Leftarrow$) $\lim d(f_n,f)=0$

$\lim \int \frac{|f_n-f|}{1+|f_n-f|} = 0$

$\lim \int |f_n-f|\,d\mu(x) = 0$

$\hookrightarrow$ # x s.t. $|f_n-f|>\varepsilon$ goes to 0

$\mu(\{x \mid |f_n-f|>\varepsilon\})<\varepsilon$

$\Rightarrow f_n\to f$ in measure ✓

(conv in L^1 $\Rightarrow$ conv. in μ)

iii.) Need $\mu(X)<\infty$, else $d\mu(x)$ not necessarily a finite measure But must be finite for the given definition to hold

5. Let $B(x,r) = \{y \in \mathbb{R}^d : |y - x| < r\}$ denote the Euclidean ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$, where $|x| = \left(\sum_{j=1}^d x_j^2\right)^{1/2}$ is the standard Euclidean norm for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. For a Lebesgue integrable function $f : \mathbb{R}^d \to \mathbb{C}$ we define

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\,dy,$$

where m denotes the d-dimensional Lebesgue measure on $\mathbb{R}^d$. Show that $A_r f(x)$ is jointly continuous in $r > 0$ and $x \in \mathbb{R}^d$, and using this deduce that the set

$$\{x \in \mathbb{R}^d : \sup_{r>0} |A_r f(x)| > \lambda\}$$

is open in $\mathbb{R}^d$ for every $\lambda > 0$.

cont from $X \times Y \to Z$

Folland Lemma 3.16
If $f \in L^1_{loc}$, $A_r f(x)$ jointly cont in $r > 0$, $x \in \mathbb{R}^n$

$m(B(x,r)) = r \cdot m(B(x,1)) = r \cdot m(B(0,1))$ by properties of Lebesgue measure

As $r \to r_0$ + $x \to x_0$, $\chi_{B(x,r)} \to \chi_{B(x_0,r_0)}$ pointwise (a.e.)

Note $|\chi_{B(r,x)}| \le \chi_{B(r_0+1,x_0)}$ ptwise if $r < r_0 + \frac{1}{2}$, $|x - x_0| < \frac{1}{2}$

DCT $\Rightarrow$ limit is continuous in r + x

so $\int_{B(r,x)} f(y)\,dy$ is jointly cont

Then so is $A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)\,dy$

SECOND PART INCOMPLETE

SPRING 2024

1. Let $\tau_h f(x) = f(x-h)$ be the translation. Find all $p \in [1,\infty]$ for which $f \in L^p(\mathbb{R})$ implies that $\lim_{h\to 0} \|f - \tau_h f\|_{L^p(\mathbb{R})} = 0$ (and justify your answer).

Folland 8.5

$\underline{\text{Sol}}$: all $p < \infty$

$\rightarrow C^\infty$ dense in L^p

let $g \in C_c^\infty$ (so g is $\underline{\text{uni cont}}$ and thus $\|g(x-h) - g(x)\|_{L^p} \to 0$ as $h \to 0$

$$\lim_{h\to 0} \|f - \tau_h f\|_{L^p}$$
$$= \lim_{h\to 0} \|f - g + g - \tau_h g + \tau_h g - \tau_h f\|_{L^p}$$
$$\le \lim_{h\to 0} \underbrace{\|f-g\|_{L^p}}_{<\varepsilon} + \underbrace{\|g - \tau_h g\|_{L^p}}_{<\varepsilon} + \underbrace{\|\tau_h g - \tau_h f\|_{L^p}}_{<\varepsilon}$$

by chosen g + dense in L^p — by uni cont — also by chosen g

$= \|\tau_h(g-f)\|_{L^p}$

$< 3\varepsilon$ let ε be arb. small, then

$\lim_{h\to 0} \|f - \tau_h f\|_{L^p} = 0$

This doesn't work for $p = \infty$. For example:

$f = \chi_{[0,1]}$

$\|f - \tau_h f\|_{L^\infty} = \|\chi_{[0,1]} - \tau_h \chi_{[0,1]}\|_{L^\infty} = 1$

true $\forall h > 0$, so $\lim_{h\to 0}$ " $= 1$

COUNTEREXAMPLE CR: DANAE

$\|f\|_{L^\infty} = \inf\{a \ge 0 \mid |f| \le a \text{ a.e.}\}$
$= \inf\{a \ge 0 \mid \mu(\{x \mid |f(x)| > a\}) = 0\}$

$\|\chi_{[0,1]} - \tau_h \chi_{[0,1]}\|_{L^\infty}$
$= \inf\{a \ge 0 \mid |\chi_{[0,1]} - \tau_h \chi_{[0,1]}| \le a \text{ a.e.}\}$
$\hookrightarrow a = 1$ ($x = 0$ or 1)

2. Let f_n $[0,1] \to \mathbb{R}$ be a sequence of measurable functions with $|f_n(x)| \leqslant 1$ for a.e. x. Let

$$g_n(x) = \int_0^x f_n(t)dt.$$

Show that there exists an absolutely continuous g and a subsequence $n_k \to \infty$ such that $g_{n_k} \to g$ in $C([0,1])$.

$f \in L^1 \Rightarrow \int f$ is abs cont
↳ 714 p5

CREDIT: BENZI

Each $g_n = \int f_n(t)\,dt$ where f_n bounded + compactly supported (only defined on $[0,1]$ so $supp(f) \subset [0,1]$, is cpt.)

Then each g_n is continuous + bounded

integral of bdd function on compact region bdd ($\int \leq M \cdot L$)

also note

$|g_n(x) - g_n(y)|_{x>y} \leq |{}_0\int^x f(t)\,dt - {}_0\int^y f(t)\,dt|$

$\leq |{}_y\int^x f(t)\,dt|$

$\leq M \cdot L = 1 \cdot (x-y) = (x-y)$

Then if $|x-y| < \varepsilon$, $|g_n(x) - g_n(y)| < \varepsilon$

So $\{g_n\}$ is equicontinuous.

note: This isn't on the qual syllabus but it has come up a few times

$\{g_n\}$ equicontinuous + pointwise bdd $\Rightarrow$ use Arzela-Ascoli

By A-A, $\exists$ a subseq. of $\{g_n\}$ which converges uniformly to $g \in C(X)$ on cpt sets.
↳ $[0,1]$ is cpt.

FTC: g abs cont $\Leftrightarrow$ $g(x) - g(a) = {}_a\int^x f(t)\,dt$ for some $f \in L^1([a,b], m)$

Let $a = 0$, then $g(x) = g(0) + {}_0\int^x f(t)\,dt$

$g(0) = {}_0\int^0 f(t)\,dt = 0$
Leb. measure of pt $= 0$

↳ $f_n : [0,1] \to \mathbb{R}$ w/ $|f_n| \leq 1$ is L^1 since ${}_0\int^x |f_n(t)|\,dt \leq M \cdot L = 1 \cdot 1 = 1 < \infty$

Then $g(x) = 0 + {}_0\int^x f(t)\,dt$

$g(x) = {}_0\int^x f(t)\,dt$ as given satisfies one of the equivalent cond. of FTC, so g is abs cont.

3. Let $f_n, g_n \in L^2([0,1])$ be sequences. Assume that for $f \in L^2([0,1])$ and $g : [0,1] \to \mathbb{R}$ measurable, $f_n \to f$ in $L^2([0,1])$ norm while $g_n \to g$ almost everywhere. Also assume that $\|g_n\|_{L^2([0,1])} \leqslant 1$. Show that the product fg is integrable, and that

$$\int f_n g_n \to \int fg.$$

$\{f_n\}, \{g_n\} \subseteq L^2$

$f_n \to f$ in L^2 norm $\qquad f \in L^2 \qquad \|f_n - f\|_{L^2} \to 0$

$g_n \to g$ a.e. $\qquad g \in L^0 \qquad |g_n(x) - g(x)| \to 0$

$\|g_n\|_{L^2} \leq 1 \qquad \|g_n\|_{L^2} \leq 1$

$\lim_{n\to\infty} \|f_n g_n\|_{L^1} \leq \lim_{n\to\infty} \underbrace{\|f_n\|_{L^2}}_{<\infty} \cdot \underbrace{\|g_n\|_{L^2}}_{\leq 1}$ (Holder)

$\lim_{n\to\infty} \|f_n g_n\|_{L^1} < \infty$ so $\in L^1$

since L^1 norm of $f_n g_n$ is $<\infty$

* ↳ also, $\exists$ some L^1 function, call it h, which bounds $f_n g_n$ $\forall n$

BENZI'S CONCLUSION:

- L^2 conv. $\Rightarrow$ ptwise conv.

  $f_n \to f$ in $L^2 \Rightarrow f_n \to f$ ptwise (a.e.)

- $g_n \to g$ a.e. (given) $\quad$ $\Big\}$ $\Rightarrow \underbrace{f_n g_n}_{\in L^1 \text{ (shown)}} \to fg$ a.e.

  $f_n \to f$ a.e. (above)

Hey, now we can use DCT!

$\{f_n g_n\} \subseteq L^1$ w/ $f_n g_n \to fg$ a.e.

& $|f_n g_n| \leq h \in L^1$

By DCT: $\lim_{n\to\infty} \int f_n g_n = \int \lim_{n\to\infty} f_n g_n$

$= \int fg$

and fg is integrable

4. Let (X, d) be a metric space, and let q be another metric on X. Assume that, for any $x \in X$, a sequence $\{x_n\} \subset X$ converges to x with respect to d if and only if it converges to x with respect to q.

(a) Show that d and q induce the same topology on X.

(b) Give an example showing that d and q need not be equivalent (*equivalent* here means that there is a constant $C > 0$ such that $C^{-1}d(x, y) \leqslant q(x, y) \leqslant Cd(x, y)$ for all $x, y \in X$).

a.) $\{x_n\} \to x$ wrt d iff $\{x_n\} \to x$ wrt q

let τ_d be the topology induced by d + τ_q induced by q

$\tau_d = \tau_q$ if all open sets under d also open under q, + vice versa

alternatively, all closed sets

Recall closed sets contain limit pts.

Let $A \subset \tau_d$. Then A open, A^c closed. Then for any $\{x_n\} \subset A^c$ s.t. $\{x_n\} \xrightarrow{d} x$, $x \in A^c$.

But $\{x_n\} \xrightarrow{d} x$ also means that $\{x_n\} \xrightarrow{q} x$. This is true for any convergent seq. in A^c, so A^c contains all limit pts wrt d

$\Rightarrow$ contains all limit pts wrt q, so A^c closed in τ_d

$\Rightarrow$ A^c also closed in τ_q.

The same proof holds taking $B \subset \tau_q$ open and considering B^c in τ_d.

Thus, closed sets are the same in τ_d + τ_q, so the induced topologies are the same

b.) $1/C\, d(x,y) \leq q(x,y) \leq C\, d(x,y)$ $\forall x,y$, some $C>0$ $\}$ contradict this w/ $\tau_d = \tau_q$

BENZI'S EXAMPLE:

$d(x,y) = |x-y|$ (Euclidean metric)

$q(x,y) = 1$ when $x \neq y$ (discrete metric)

$\{x_n\} \to x \Rightarrow d(x_n,x) = q(x_n,x) \to 0$

$d(x_n,x) \to 0 \Rightarrow |x_k - x| < \varepsilon$ $\forall k \geq N$ for some N

$q(x_n,x) \to 0 \Rightarrow x_k = x$ $\forall k \geq N$ for some N

make these be equivalent by letting $X = \mathbb{N}$ (then $d(x,y) \in \mathbb{N}$ $\forall x,y$, so $|x_k - x| < \varepsilon \Rightarrow |x_k - x| = 0$)

so convergence is equiv, + thus topo is equiv,

but $\nexists C>0$ s.t. $1/C\, d(x,y) \leq q(x,y) \leq C\, d(x,y)$ $\forall x,y$

$\boxed{\text{e.g.}}$ $x \neq y$, $q(x,y) = 1$ but $d(x,y) < \varepsilon$ can be arbitrarily small (so no C satisfies)

5. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of Lebesgue integrable functions with $f_n \to f$ in $L^1([0,1])$.

(a) Define, for any $t \in \mathbb{R}$, the function

$$g(x) = \begin{cases} 1 & x > t \\ 0 & x \leqslant t. \end{cases}$$

Show that there exists a subsequence n_k such that

$$g \circ f \leqslant \liminf_{k \to \infty} g \circ f_{n_k}$$

almost everywhere.

(b) Show that, for any $t \in \mathbb{R}$,

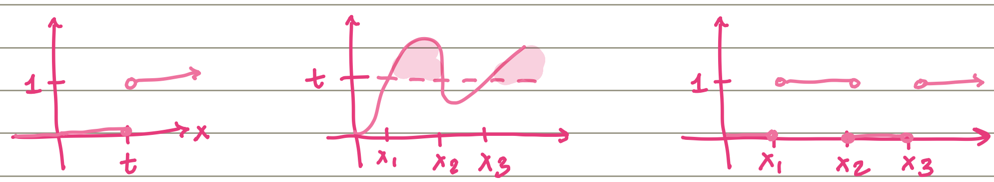
$$|\{x \in [0,1] : f(x) > t\}| \leqslant \liminf_{n \to \infty} |\{x \in [0,1] : f_n(x) > t\}|.$$

(c) Show that for all but countably many t, $\lim_{n\to\infty} |\{x \in [0,1] : f_n(x) > t\}|$ exists and equals $|\{x \in [0,1] : f(x) > t\}|$.

CR: MARK

a.) a function is lower semicontinuous at $x_0 \in X$ if for every $y \in \mathbb{R}$ w/ $y < f(x_0)$, $\exists$ nbhd U of x_0 s.t. $f(x) > y$ $\forall x \in U$.

↳ Equiv: f is lower semicont @ x_0 iff $\liminf_{x \to x_0} f(x) \geq f(x_0)$



$g(x)$ $f(x)$

$$g \circ f_n(x) = \begin{cases} 1 & f_n(x) > t \\ 0 & f_n(x) \leq t \end{cases}$$

Notice that g is lower semicont. Then $g(x_0) \leq \liminf_{x \to x_0} g(x)$

Since $f_n \to f$ in L^1, $\exists$ subseq. $\{n_k\}$ s.t. $f_{n_k} \to f$ a.e.

Then, for a.e. x : $g \circ f(x) \leq \liminf_{k \to \infty} g \circ f_{n_k}(x)$ ✓

b.) $\{x \in [0,1] \mid f(x) > t\} = \{x \mid g \circ f(x) = 1\}$

$| \text{ " } | = \int_0^1 g \circ f(x)\,dx$

f_n, g are leb intable. Then $g \circ f_n$ also intable (so this is allowed)

and $g \circ f_n \in L^+$ since $g \circ f_n(x) = 0$ or 1 only

Can use Fatou:

$\int \liminf g \circ f_n \leq \liminf \int g \circ f_n$

By part (a.), $g \circ f \leq \liminf g \circ f_{n_k}$, so $\int g \circ f \leq \int \liminf g \circ f_{n_k}$

By the characterization of liminf, $\exists$ subseq. of f_n conv. to $\liminf f_n$

So $f_{n_k} \to \liminf f_n$

$\Rightarrow \liminf f_{n_k} = \lim f_{n_k} = \liminf f_n$

Put it all together: $\int g \circ f \leq \int \liminf g \circ f_{n_k}$

$= \int \liminf g \circ f_n$

$\leq \liminf \int g \circ f_n$

$\Rightarrow |\{x \mid f(x) > t\}| \leq \liminf |\{x \mid f_n(x) > t\}|$ ✓

c.) monotone functions are cont. up to countable sets ↗(as t increases)

Define $h(t) = \{x \mid f(x) > t\}$. Note that $h(t)$ is monotone decreasing.
Then $h(t)$ is cont up to a countable set. (Then at most countably many pts don't satisfy.)

WTS: $\lim_{n\to\infty} |\{x \mid f_n(x) > t\}| = |\{x \mid f(x) > t\}|$ at all pts of continuity.

let t be a pt of continuity. Then from (b.):

$|\{x \mid f(x) > t\}| \le \liminf |\{x \mid f_n(x) > t\}|$

$|\{x \mid f(x) > t\}| \le |\{x \mid f_n(x) > t\}| + \varepsilon$

Note: $|\{x \mid (f+g)(x) > a+b\}| \le |\{x \mid f(x) > a\}| + |\{x \mid g(x) > b\}|$

So $|\{x \mid f_n(x) > t\}| \le |\{x \mid (f_n - f)(x) > \delta\}| + |\{x \mid f(x) > t-\delta\}|$ for some $\delta > 0$

or $|\{x \mid f_n(x) > t\}| \le |\{x \mid (f_n - f)(x) > \delta\}| + |h(t-\delta)|$

" $\le$ " $+ |h(t-\delta) - h(t) + h(t)|$

" $\le$ " $+ |h(t-\delta) - h(t)| + |h(t)|$

or $|\{x \mid f_n(x) > t\}| - \underbrace{|h(t-\delta) - h(t)|}_{<\varepsilon \text{ by cont. of } h} - |\{x \mid (f_n - f)(x) > \delta\}| \le |h(t)|$

Now take suff. large N s.t. whenever $n \ge N$, $|\{(f_n - f)(x) > \delta\}| < \varepsilon$

Then whenever $n \ge N$:

$|\{f_n(x) > t\}| - \varepsilon - \varepsilon \le |\{f(x) > t\}| \le |\{f_n(x) > t\}| + \varepsilon$

Then for suff large N, as $\varepsilon \to 0$, $|\{f_n(x) > t\}| = |\{f(x) > t\}|$

thus $\lim |\{f_n(x) > t\}| = |\{f(x) > t\}|$ ✓

FALL 2024

If X is a topological space and $\mathcal{F} \subset C(X)$, $\mathcal{F}$ is called **equicontinuous at $x \in X$** if for every $\epsilon > 0$ there is a neighborhood U of x such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and all $f \in \mathcal{F}$, and $\mathcal{F}$ is called **equicontinuous** if it is equicontinuous at each

CR: TIM

1. Let (X, ρ) be a metric space.

has a ctable, dense subset

(a) Suppose that (X, ρ) is separable. Prove that if Y is any subset of X, then (Y, ρ) is separable.

(b) Suppose that (X, ρ) is compact. Let $\mathcal{F}$ be any set of real valued functions on X that is uniformly bounded and equicontinuous. Is the function

$$g(x) = \sup\{ f(x) \ : \ f \in \mathcal{F} \}$$

necessarily continuous? Prove that your answer is correct.

a.) let $\{x_n\}$ be a countable, dense subset of X. Then $\{B_{1/m}(x_n)\}_{m,n}$ is a ctable basis for the topology on X.

claim: $\{Y \cap B_{1/m}(x_n)\}_{n,m}$ is a basis for Y

* $U \subseteq Y$ is open iff $\exists$ open $V \subseteq X$ s.t. $U = V \cap Y$.
Since $\{B_{1/m}(x_n)\}$ is a basis for X, let $V := \bigcup_{i=1}^{\infty} B_{1/m_i}(x_{n_i})$
so that $U := \bigcup_{i=1}^{\infty} Y \cap B_{1/m_i}(x_{n_i})$

Then $\{B_{1/m}(x_n)\}$ is a basis for Y, too.

Now, let $\{y_{n,m}\} \subseteq Y$ s.t. $y_{n,m} \in Y \cap B_{1/m}(x_n)$ $\forall m,n$
(whenever intersection is nonempty)

Observe that $\{y_{n,m}\}$ is a countable (duh) dense subset of Y

If $y \in Y$, then for any $\varepsilon > 0$, $B_\varepsilon(y) = \bigcup_i B_{1/m_i}(x_{n_i}) \cap Y$
Then for some i, $d(y, y_{m_i,n_i}) < \varepsilon$ ✓

b.) $\mathcal{F}$ uni bdd. $\Rightarrow |g(x)| < \infty$ $\forall x$, so g well-defined

Let $\varepsilon > 0$. As $\mathcal{F}$ is equicont, $\exists \delta > 0$ s.t. $\rho(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ $\forall f \in \mathcal{F}$.

Fix x & y w/ $\rho(x,y) < \delta$, then $\exists f_1, f_2 \in \mathcal{F}$ s.t. $|g(x) - f_1(x)| < \varepsilon$
$|g(x) - f_2(x)| < \varepsilon$

$$\begin{aligned}
\Rightarrow |g(x) - g(y)| &= |g(x) - f_1(x) + f_1(x) - f_2(y) + f_2(y) - g(y)| \\
&\le \underbrace{|g(x) - f_1(x)|}_{<\varepsilon} + |f_1(x) - f_2(y)| + \underbrace{|f_2(y) - g(y)|}_{<\varepsilon} \\
&< 2\varepsilon + |f_1(x) - f_2(y)| \\
&= 2\varepsilon + |f_1(x) - f_1(y) + f_1(y) - f_2(y)| \\
&\le 2\varepsilon + \underbrace{|f_1(x) - f_1(y)|}_{<\varepsilon} + |f_1(y) - f_2(y)|
\end{aligned}$$

$$\Rightarrow |g(x) - g(y)| < 3\varepsilon + |f_1(y) - f_2(y)|$$

INCOMPLETE

but the answer is yes

2. Let μ be Lebesgue measure on $\mathbb{R}$.

(a) Let
$$A := \left\{ f \in L^1(\mu) \ : \ \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}\mu \geqslant 1 \right\}.$$
Is A a closed subset of $L^1(\mu)$? Prove that your answer is correct.

(b) Let
$$B := \left\{ f \in L^1(\mu) \ : \ \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}\mu \leqslant 1 \right\}.$$
Is B a closed subset of $L^1(\mu)$? Prove that your answer is correct.

See Fall 15 Prob. #1

3. Let μ denote Lebesgue measure on $[0,1]$. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $[0,1]$ such that
$$\int_{[0,1]} |f_n(x)|^2 \, \mathrm{d}\mu(x) \leqslant 1 \quad \text{for all } n.$$
Suppose that the sequence $\{f_n\}$ converges to zero in measure. Show that
$$\lim_{n\to\infty} \int_{[0,1]} f_n(x) \, \mathrm{d}\mu(x) = 0 .$$

$\mu(\{x \mid |f_n(x) - 0| \geq \varepsilon\}) \to 0$

Since $f_n \xrightarrow{\mu} 0$, $\forall \varepsilon > 0 \ \exists N$ s.t. $\forall n \geq N$, $\mu(\{x \mid |f_n(x)| \geq \varepsilon\}) < \varepsilon$.

Let $\varepsilon > 0$, and let $E_1 = \{x \mid |f_n(x)| \geq \varepsilon\}$ so $[0,1] = E_1 \cup E_2$
(fixing n) $\quad E_2 = \{x \mid |f_n(x)| < \varepsilon\}$

$\int_{[0,1]} f_n(x)\,dx = \int_{E_1} f_n(x)\,dx + \int_{E_2} f_n(x)\,dx$

Note that if $\lim \int |f_n| = 0$, then $\lim \int f_n = 0$
so we will actually proceed w/ the stronger claim

$\int_{E_1} |f_n(x)|\,dx = \|f_n\|_{L^1(E_1)} \leq \|f_n\|_{L^2(E_1)} \cdot \|1\|_{L^2(E_1)}$ by Holder's ineq.

$= \left(\int |f_n|^2 dx\right)^{1/2} \cdot 1 \ \leq \ \underset{(m \cdot L)}{\left(1 \cdot \mu(E_1)\right)^{1/2}} \cdot 1 \ = \ \mu(E_1)^{1/2}$

$\int_{E_2} |f_n(x)|\,dx \leq \underset{(m \cdot L)}{\varepsilon \cdot \mu(E_2)}$

Then $\lim \int |f_n(x)|\,dx \leq \lim \left(\mu(E_1)^{1/2} + \mu(E_2) \cdot \varepsilon\right)$
$= \lim \mu(E_1)^{1/2} + \lim \mu(E_2) \cdot \varepsilon$
$= \quad 0 \quad + \mu([0,1]) \cdot \varepsilon$
$= \quad 0 \quad + \ 1 \cdot \varepsilon$
$= \varepsilon$ ✓

Sending $\varepsilon \to 0$, we get that $\lim \int_{[0,1]} |f_n(x)|\,dx = 0$, so
$\lim \int_{[0,1]} f_n(x)\,dx = 0$ ✓

4. Let μ^* denote Lebesgue outer measure on $\mathbb{R}$. Let A and B be any two subsets of $\mathbb{R}$ that are separated by a positive distance d. That is, if $x \in A$ and $y \in B$, then $|x-y| \geqslant d > 0$. Show that

metric space

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Subadditivity of outer measure gives us that

$\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$

Then it suffices to demonstrate the other inequality

Let $\varepsilon > 0$. By definition of outer measure:

$\mu^*(A) = \inf \{ \sum_{i=1}^{\infty} \mu(E_i) \mid \bigcup_{i=1}^{\infty} E_i \supset A \}$

$\mu^*(B) = \inf \{ \sum_{i=1}^{\infty} \mu(F_i) \mid \bigcup_{i=1}^{\infty} F_i \supset B \}$

Since $d(A,B) = D > 0$, there exist disjoint collections $\{E_i\}_{i=1}^{\infty}$ + $\{F_i\}_{i=1}^{\infty}$ s.t. $A \subset \bigcup_{i=1}^{\infty} E_i$ + $B \subset \bigcup_{i=1}^{\infty} F_i$

Then:

$$\begin{aligned} \mu^*(A) + \mu^*(B) &\leq \mu^*(\textstyle\bigcup_{i=1}^{\infty} E_i) + \mu^*(\bigcup_{i=1}^{\infty} F_i) \\ &\leq \textstyle\sum_{i=1}^{\infty} \mu^*(E_i) + \sum_{i=1}^{\infty} \mu^*(F_i) \\ &= \textstyle\sum_{i=1}^{\infty} \mu^*(E_i) + \mu^*(F_i) \\ &\leq \mu^*(A \cup B) + \varepsilon \end{aligned}$$

by subadd. of μ^*

Since ε was arbitrary, letting $\varepsilon \to 0$ gives us that

$\mu^*(A) + \mu^*(B) \leq \mu^*(A \cup B)$

so we have the other inequality.

5: Let μ be Lebesgue measure on $\mathbb{R}$. For any $f \in L^1(\mu)$ and any $a > 0$, define $f_a(x) := af(ax)$.

(a) Prove that for all $a > 0$, $\int_{\mathbb{R}} f_a(x)\mathrm{d}\mu = \int_{\mathbb{R}} f(x)\mathrm{d}\mu$.

(b) Prove that

$$\lim_{a\to 1} \int_{\mathbb{R}} |f(x) - f_a(x)|\mathrm{d}\mu = 0 \text{ .}$$

a.) $_{\mathbb{R}}\int f_a(x)\,dx = {}_{\mathbb{R}}\int a\cdot f(ax)\,dx = a\,_{\mathbb{R}}\int f(ax)\,dx = a\,_{\mathbb{R}}\int f(y)\cdot\frac{1}{a}\,dy$

let $y = ax$, $dy = a\cdot dx$ $\quad = {}_{\mathbb{R}}\int f(y)\,dy$ ✓

CR: OWEN

b.) $f \in L^1$ so by part (a.), $f_a(x) \in L^1$.

Define $f_n(x) = \frac{n-1}{n} f\left(\frac{n-1}{n}\cdot x\right)$

Then note:

- $f_n(x) \to f(x)$ a.e.
- $\{f_n(x)\} \subseteq L^1$
- $\lim_{n\to\infty} f_n(x) = \lim_{a\to 1} f_a(x) = f(x)$

(setting up DCT)

$\lim f_n(x) = \lim \frac{n-1}{n} f\left(\frac{n-1}{n}\cdot x\right) = 1\cdot f(1\cdot x) = f(x)$

$\frac{n-1}{n} > 0$, so $\in L^1$ by pt (a.)

and $|f(x) - f_a(x)| \leq |f(x)| + |f_a(x)| \leq \underbrace{2\cdot\max\{f_n(x), f(x)\}}_{g(x)} \in L^1$

Then we can apply DCT:

$\lim_{a\to 1}\int|f(x) - f_a(x)|\,dx$
$= \lim_{n\to\infty}\int|f(x) - f_n(x)|\,dx$
$= \int\lim_{n\to\infty}|f(x) - f_n(x)|\,dx$
$= \int 0$
$= 0$ ✓

SPRING 2025

Problem 1. Let f, g be absolutely continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and $h = f \circ g$ be their composition.

Prove that if g is nondecreasing, then h is absolutely continuous.

absolutely continuous on $\mathbb{R}$ = $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for any finite set of disjoint intervals $\{(a_i, b_i)\}_{i=1}^N$ w/ $(a_i, b_i) \subset \mathbb{R}$,

$$\sum_{i=1}^N |b_i - a_i| < \delta \implies \sum_{i=1}^N |f(b_i) - f(a_i)| < \varepsilon$$

Let $\{(a_i, b_i)\}_{i=1}^N$ be a disjoint collection of intervals s.t. $(a_i, b_i) \subset \mathbb{R}$.

Let $\varepsilon > 0$. The absolute continuity of f gives us a $\delta' > 0$ and any finite collection of intervals $\{(c_i, d_i)\}_{i=1}^N$ such that $\sum_{i=1}^N |d_i - c_i| < \delta'$. Since g is nondecreasing, $\{g^{-1}(c_i, d_i)\}_{i=1}^N$ is also a finite collection of disjoint intervals in $\mathbb{R}$. Let $g^{-1}(c_i, d_i) = (a_i, b_i)$ $\forall i$. Then using the absolute continuity of g, $\exists \delta > 0$ corresponding to δ' s.t. whenever $\sum_{i=1}^N |b_i - a_i| < \delta$, $\sum_{i=1}^N |g(b_i) - g(a_i)| < \delta'$.
all together, we have:

$$\sum_{i=1}^N |b_i - a_i| < \delta \implies \sum_{i=1}^N |g(b_i) - g(a_i)| < \delta'$$

$$\implies \sum_{i=1}^N |d_i - c_i| < \delta' \implies \sum_{i=1}^N |f(d_i) - f(c_i)| < \varepsilon$$

$$\sum_{i=1}^N |h(b_i) - h(a_i)| = \sum_{i=1}^N |f(g(b_i)) - f(g(a_i))| = \sum_{i=1}^N |f(d_i) - f(c_i)| < \varepsilon \checkmark$$

So h is absolutely continuous!

Problem 2.
Let dx denote Lebesgue measure on the Lebesgue σ-algebra of $\mathbb{R}$. Assume $f \in L^1(\mathbb{R}, dx)$ is real-valued. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) := \int_{\mathbb{R}} f(y) e^{-(x-y)^2} dy.$$

Prove that the map $x \mapsto g(x)$ is Lipschitz continuous on $\mathbb{R}$.

$\exists M$ s.t.
$|f(x) - f(y)| \le M|x-y|$

$$|g(x_1) - g(x_2)| = \left| \int f(y) e^{-(x_1-y)^2} dy - \int f(y) e^{-(x_2-y)^2} dy \right|$$
$$= \left| \int f(y) \left(e^{-(x_1-y)^2} - e^{-(x_2-y)^2}\right) dy \right|$$
$$\le \int |f(y)| \cdot \left| e^{-(x_1-y)^2} - e^{-(x_2-y)^2} \right| dy$$

$$\int |f(y)| \cdot \underbrace{\left| e^{-(x_1-y)^2} - e^{-(x_2-y)^2} \right|}_{\le M|x_1 - x_2|} dy \le M|x_1 - x_2| \underbrace{\int |f(y)| dy}_{<\infty} = CM|x_1 - x_2| \checkmark$$

$\hookrightarrow$ assumes e^{-x^2} is Lipschitz?

$<\infty$ since $f \in L^1(\mathbb{R})$.
Say $\|f\|_{L^1} = C$

INCOMPLETE!

Folland Ch. 2 Exercise 46

Problem 3.
Let $X = Y = [0,1]$, both equipped with the Borel σ-algebra $\mathcal{B}([0,1])$. Let λ denote Lebesgue measure restricted to $\mathcal{B}$. Let ν denote the counting measure: for $A \subset [0,1]$, and $\mathcal{N} \subset A$ finite, $\nu(A) = \sup_{\mathcal{N}\subset A}\{\text{number of elements in}\mathcal{N}\}$. Let $D := \{(x,y) \in X \times Y : x = y\}$. Let χ_D denote the characteristic function of the set D, which may be seen as a function of x with y a parameter, or a function of y with x a parameter, or a function of (x,y) — depending on the context.

Show that the three integrals

$$I_1 := \int_Y \left(\int_X \chi_D(x)d\lambda(x)\right) d\nu(y)$$

$$I_2 := \int_X \left(\int_Y \chi_D(y)d\nu(y)\right) d\lambda(x)$$

$$I_3 := \int_{X\times Y} \chi_D(x,y)d(\lambda(x)\times\nu(y))$$

yield three different results, and [explain (briefly) why this is not a counter-example to the Fubini-Tonelli theorem.]

Fubini-Tonelli requires that $(X,\mathcal{B},\lambda)$ and $(Y,\mathcal{B},\nu)$ are both σ-finite measure spaces.
However, the counting measure ν is not σ-finite on $[0,1]$ since $[0,1] \subset \mathbb{R}$ is uncountable

$$I_1 = \int_y \left(\int_x \chi_D(x)\,d\lambda\right) d\nu = \int_y \underbrace{\lambda(D_x)}_{\substack{\text{fixed } y,\text{ measure } D\\ \text{(only 1 pt., where } x=y)}} d\nu = \int_y 0\,d\nu = 0$$

$$I_2 = \int_x \left(\int_y \chi_D(y)\,d\nu\right) d\lambda = \int_x \underbrace{\nu(D^y)}_{\substack{\text{fixed } x,\text{ count } y\in D\\ \text{(1 pt, where } y=x)}} d\lambda = \int_x 1\,d\lambda = \lambda(X) = 1$$

counting measure picks up ∞ many pts on $[0,1]$

$$I_3 = \int_{X\times Y} \chi_D(x,y)\,d(\lambda(x)\times\nu(y)) = (\lambda\times\nu)(D(x,y)) = 1\times\infty = \infty \checkmark$$

Problem 4.
Let dx denote Lebesgue measure on the Lebesgue σ-algebra of $\mathbb{R}$.
Let $\{f_n : \mathbb{R} \to \mathbb{C};\ n \in \mathbb{N}\}$ be a sequence of functions with uniformly bounded $L^2(\mathbb{R}, dx)$-norm. Suppose that there is an $f \in L^2$ such that for any $h \in L^2$ one has that $\int_{\mathbb{R}} f_n(x)h(x)dx \to \int_{\mathbb{R}} f(x)h(x)dx$ when $n \to \infty$. Suppose furthermore that the sequence $\{g_n : \mathbb{R} \to \mathbb{C};\ n \in \mathbb{N}\} \subset L^2(\mathbb{R}, dx)$ converges in L^2 topology to g.

Show that $\int_{\mathbb{R}} f_n(x)g_n(x)dx \to \int_{\mathbb{R}} f(x)g(x)dx$ as $n \to \infty$.

(HINT: The stated assumptions do **not** imply that $f_n \to f$ in L^2 topology.)

$\{f_n\}$ uni bdd L^2 norm
$(\|f_n(x)\|_{L^2} \leq M$ for some M, $\forall x)$ $\forall n$

$\exists f(x) \in L^2$ s.t. $\forall h(x) \in L^2$
$\int f_n h \to \int f h$

$g_n \to g$ in L^2

WTS: $\left|\int f_n(x) g_n(x) - \int f(x) g(x)\right| \to 0$

$\lim\left|\int f_n(x) g_n(x) - \int f(x) g(x)\right| = \lim\left|\int f_n(x) g_n(x) - f(x) g(x)\right|$

$= \lim\left|\int f_n g_n - f_n g + f_n g - f g\right|$

$\leq \lim\left(\left|\int f_n g_n - f_n g\right| + \left|\int f_n g - f g\right|\right) \leq \lim\left(\int |f_n g_n - f_n g| + \left|\int f_n g - f g\right|\right)$

$\lim \int |f_n g_n - f_n g| = \lim \|f_n g_n - f_n g\|_{L^1} = \lim \|f_n(g_n - g)\|_{L^1}$

By Hölder's ineq:

$\lim \|f_n(g_n - g)\|_{L^1} \leq \lim\left(\|f_n\|_{L^2} \cdot \|g_n - g\|_{L^2}\right) = \lim \underbrace{\|f_n\|_{L^2}}_{\leq M} \cdot \underbrace{\|g_n - g\|_{L^2}}_{=0} = 0$

$g_n \to g$ in L^2

$\lim\left|\int f_n g - f g\right| = 0$ since $\int f_n h \to \int f h$ for any $h \in L^2$

$\Rightarrow \lim\left|\int f_n g_n - \int f g\right| \leq 0 + 0 = 0$ ✓

Problem 5.
Let $X = [0, 1]$, and let dx denote Lebesgue measure on X.

(a) Let $F : \mathbb{R} \to \mathbb{R}$ be the periodic odd function of period 2 that takes the value 1 for $x \in (0, 1)$ and the value 0 at $x = 1$. (Note that this completely defines F.) Now consider the sequence $\{f_n; n \in \mathbb{N}\}$ with $f_n(x) := F(2^{n-1}x)$ restricted to X.

Show that when $n \to \infty$, then $\int_X g(x)f_n(x)dx \to 0$ for each $g \in L^2(X, dx)$, but that $f_n \nrightarrow 0$ in $L^2(X, dx)$ topology, and that $f_n \nrightarrow 0$ pointwise a.e. in X, and that $f_n \nrightarrow 0$ in measure either.

(HINT: It helps to graph the first few members of the sequence $\{f_n\}$, say for $n \in \{1, 2, 3\}$. It also helps to recall Bessel's inequality.)

(b) Consider the sequence $\{f_n; n \in \mathbb{N}\}$, with $f_n(x) := n\chi_{[0,1/n)}(x)$.

Show that when $n \to \infty$, then $f_n \to 0$ pointwise a.e. in X, and $f_n \to 0$ also in measure, but $f_n \nrightarrow 0$ pointwise in X, and also $f_n \nrightarrow 0$ weakly in $L^2(X, dx)$. (Hint: For the last task one of these four tasks it suffices to find a suitable $g \in L^2(X, dx)$ such that $\int_X n\chi_{[0,1/n)}(x)g(x)dx \nrightarrow 0$ when $n \to \infty$.)