THE COMBINATORICS OF LANE MERGING (RECURSIVE APPROACH)

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In this note, we provide an alternative, recursive approach to the proof of Theorem 1 in "The Combinatorics of Lane Merging" (2026) by Hiveley and Zeilberger.

Two- and Three-Lane Case. The recurrence for two lane arrival sequences has previously been studied in [`?bard`], but we include it here as well for the sake of completeness, and to show how the base calculation extends to additional lanes.

Proposition 1. *The recurrence for a 2-lane merging sequence with x cars in the right lane and y cars in the left lane is:*

$$F(x,y) = \begin{cases} F(x,y-1) + F(x-1,y) & x-y>1 \\ F(x,y-1) + 2F(x-1,y) & x-y=1 \\ F(x,y-1) & x=y \end{cases}$$

Noting that $F(x,y)=0$ if $y>x$, and $F(0,0)=1$.

This proposition is equivalent to Lemma 5 of Bardenova et al, and the author refers the reader to the proof in their paper. Our work begins by adapting this recurrence to fit arrival sequences with three lanes.

Proposition 2. *The recurrence for a 3-lane merging sequence with x cars in the right lane, y cars in the middle lane, and z cars in the left lane is:*

$$F(x,y,z) = \begin{cases} F(x-1,y,z) + F(x,y-1,z) + F(x,y,z-1) & x-y>1 \text{ and } y-z>1 \\ 2F(x-1,y,z) + F(x,y-1,z) + F(x,y,z-1) & x-y\le 1 \text{ and } y-z>1 \\ F(x-1,y,z) + 2F(x,y-1,z) + F(x,y,z-1) & x-y>1 \text{ and } y-z\le 1 \\ 2F(x-1,y,z) + 2F(x,y-1,z) + F(x,y,z-1) & x-y=1 \text{ and } y-z=1 \\ 3F(x-1,y,z) + 2F(x,y-1,z) + F(x,y,z-1) & o.w. \end{cases}$$

Noting that $F(x,y,z)=0$ if $z>y$ or $y>x$, and $F(0,0,0)=1$.

We will defer our proof of this proposition and instead direct the reader towards the proof of the more general recurrence in the next section. In the meantime, we offer the following examples:

Example 1. If the ending configuration of our three lane arrival sequence is $(4,2,1)$, then there are three possible lanes which the seventh (last) car could have joined:

(1) If Car 7 joined the right lane, then the previous configuration would have been $(3,2,1)$. This outcome occurs if Car 7's preference is a 1, as any other preference would cause the car to pick a shorter lane.
(2) If Car 7 joined the middle lane, then the previous configuration would have been $(4,1,1)$. This outcome occurs if the car's preference was either a 2 or a 3. If the car's preference was a 2, then the car chooses the shorter of the right and middle lanes, which is, of course, the middle lane. If the car's preference was a 3, then it picks the shortest of all lanes, opting for the rightmost lane in case of a tie. Then the middle lane is selected instead of the left, even though their lengths are the same.
(3) If Car 7 joined the left lane, then the previous configuration would have been $(4,2,0)$. The only preference which produces this outcome is a 3.

Then the overall recurrence is $F(4,2,1) = F(3,2,1) + 2F(4,1,1) + F(4,2,0)$.

Example 2. If the ending configuration of our three lane arrival sequence is $(3,3,2)$, then the final car can only have joined two of the possible lanes: the middle or the left. If the final car joined the right lane, that would mean that the previous configuration was $(2,3,2)$, which is not possible since the lane lengths must be weakly decreasing from right to left. Then we only have two cases:

(1) If the final car joins the middle lane, then the previous configuration was $(3,2,2)$. Then Car 8's preference could have been either a 2 or a 3, and in either case the car would have chosen the middle lane, either by breaking a tie or by selecting the shorter of two lanes.
(2) If the final car joins the left lane, then the previous configuration was $(3,2,1)$. This means that Car 8's preference must have been a 3.

This means that $F(3,3,2) = 2F(3,2,2) + F(3,2,1)$.

The reader may also notice that the coefficient of each term of the recurrence is equal to the size of the "bounce" taken when the next car joins the indicated lane. In this sense, the coefficients arise quite naturally from the visualizations in the previous section, as we can simply count how many rows of the tableau have the same length as the row that we will append n to.

Generalized Recurrence. We now generalize the recurrences developed in the previous section to a general number of lanes, k.

Theorem 3. *The general recurrence for a k-lane arrival sequence with lane lengths $(x_1, x_2, \ldots, x_k)$ is:*

$$F(x_1, \ldots, x_k) = \sum_{i=1}^{k} c_i \cdot F(x_1, \ldots, x_i - 1, \ldots, x_k)$$

where c_i is defined:

$$c_i := 1 + \#\{j > i \mid x_j = x_i - 1\}$$

and $F(0, \ldots, 0) = 1$, and $F(x_1, \ldots, x_k) = 0$ if $x_i < x_j$ for any $1 \le i < j \le k$.

Example 3. $F(4,3,3,2) = 3F(3,3,3,2) + 2F(4,3,2,2) + F(4,3,3,1)$

(1) First, $c_1 = 1 + 2 = 3$ since x_2 and x_3 are equal to $x_1 - 1 = 3$.
(2) Secondly, note that $F(3,2,3,2) = 0$ since $x_2 < x_3$. So this term does not appear in the recurrence.
(3) Thirdly, $c_3 = 1 + 1$ since $x_4 = x_3 - 1 = 2$. So $j = 4$ is the only entry in the set counted by c_3.
(4) Lastly, $c_4 = 1 + 0 = 1$ since there are no $j > 4$ to be considered in the set $\{j > i \mid x_j - x_i - 1\}$.

Proof. Of course, there is one sequence of length zero (the empty sequence), so $F(0, \ldots, 0) = 1$. Furthermore, there can never be more cars in a left-er lane than in the lanes to the right, as this would violate the rules of a tie. This means that $F(x_1, \ldots, x_k) = 0$ if $x_i < x_j$ for any $i < j$.

We will follow the same argument used in the proof of Lemma 1 from the main paper. Say that the end state of an arrival sequence with k lanes and n cars is $(x_1, \ldots, x_k)$, and that deleting the n-th car from the sequence removes a car from lane i. There are $F(x_1, \ldots, x_i - 1, \ldots, x_k)$ arrival sequences which produce the previous endstate before Car n merged. The number of preferences that Car n could have had which would have placed it in lane i is equal to the number of lanes to the left of i which are tied with lane i before Car n is resolved, which is counted by c_i. Then to count all arrival sequences which produce the desired end state, we must consider all k possible lanes which Car n could have joined, hence we sum this quantity over all possible lanes $1 \le i \le k$ that Car n could have joined, and we obtain the desired recurrence. □

Corollary 4. *The general recurrence for a k-candidate majority voting sequence with end result $(p_1, p_2, \ldots, p_k)$ (from most to least popular) is given by the recurrence in Theorem 3.*

Proof. Follows immediately from Lemma 5 of the original paper. □

Knowing that arrival sequences and majority voting sequences have the same recurrence and initial conditions allows us to conclude that they are counted identically, and so a closed form that describes the number of majority voting sequences which produce the end state $(p_1, \ldots, p_k)$ will also describe the number of arrival sequences with the same end state. Such a closed form is given in Lemma 7 of the original paper.

Corollary 5. *Arrival sequences are counted by the same closed form that voting sequences satisfy.*

Proof. Since both arrival sequences and majority walks satisfy the same recurrence with the same initial conditions, both objects are counted the same way. Then the closed form for majority walks also functions as a closed form for arrival sequences. □

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