Analyzing Permutation Wordle

Graduate Student Combinatorics Conference 2026

Aurora Hiveley

Department of Mathematics
Rutgers University

Permutation Wordle

1. I'm thinking of $\pi \in \mathcal{S}_n$
2. Your goal: guess π in as few guesses as possible
3. After each guess, I will tell you which indices (if any) are correct
4. The game ends when you have guessed my permutation

Example

12345

Example

1**2**345

Example

1**2**345

52134

Example

1**2**345

5**2**1**3**4

Example

1**2**345

5**2**1**3**4

42531

Example

1**2**345

5**2**1**3**4

42531

Example

$$\gamma_1 = \quad 1\mathbf{2}345$$

$$\gamma_2 = \quad 5\mathbf{2}1\mathbf{3}4$$

$$\gamma_3 = \quad \mathbf{42531}$$

Example

$$\gamma_1 = \quad 1\mathbf{2}345 \qquad \begin{cases} \mathcal{I}_1 = \{1,3,4,5\} \\ \mathcal{J}_1 = \{2\} \end{cases}$$

$$\gamma_2 = \quad 5\mathbf{2}1\mathbf{3}4 \qquad \begin{cases} \mathcal{I}_2 = \{1,3,5\} \\ \mathcal{J}_2 = \{2,4\} \end{cases}$$

$$\gamma_3 = \quad \mathbf{42531} \qquad \begin{cases} \mathcal{I}_3 = \emptyset \\ \mathcal{J}_3 = \{1,2,3,4,5\} \end{cases}$$

Cyclic Shift Conjecture

Def. In *cyclic shift* (abbreviated CS):

Cyclic Shift Conjecture

Def. In *cyclic shift* (abbreviated CS):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \dots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \dots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

Cyclic Shift Conjecture

Def. In *cyclic shift* (abbreviated CS):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \ldots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \ldots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

Example.

Cyclic Shift Conjecture

Def. In *cyclic shift* (abbreviated *CS*):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \ldots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \ldots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

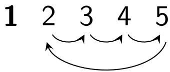
Example.

1 2 3 4 5

Def. In *cyclic shift* (abbreviated *CS*):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \dots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \dots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

Example.



Def. In *cyclic shift* (abbreviated *CS*):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \ldots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \ldots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

Example.

$$\mathbf{1}\ 2\ 3\ 4\ 5 \quad \Longrightarrow \quad \mathbf{1}\ 5\ \mathbf{2}\ 3\ 4$$

Cyclic Shift Conjecture

Def. In *cyclic shift* (abbreviated *CS*):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \ldots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \ldots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

Example.

$$\mathbf{1}\ \ 2\ \ 3\ \ 4\ \ 5 \quad \Longrightarrow \quad \mathbf{1}\ \ 5\ \ \mathbf{2}\ \ 3\ \ 4$$

Cyclic Shift Conjecture

Def. In *cyclic shift* (abbreviated *CS*):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \ldots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \ldots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

Example.

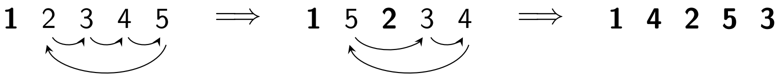
$$\mathbf{1}\ 2\ 3\ 4\ 5 \implies \mathbf{1}\ 5\ \mathbf{2}\ 3\ 4 \implies \mathbf{1}\ \mathbf{4}\ \mathbf{2}\ \mathbf{5}\ \mathbf{3}$$

Cyclic Shift Conjecture

Def. In *cyclic shift* (abbreviated *CS*):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \ldots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \ldots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

Example.



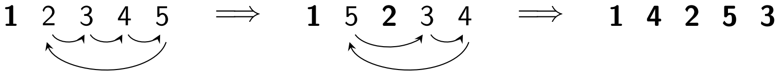
Conjecture. *CS* attains the maximal probability of success in r guesses

Cyclic Shift Conjecture

Def. In *cyclic shift* (abbreviated *CS*):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \ldots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \ldots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

Example.



Conjecture. *CS* attains the maximal probability of success in r guesses

Alt: *CS* has the smallest average number of guesses for $\pi \in \mathcal{S}_n$

Strategy Formalization

Strategy Formalization

- $S := [S_1, S_2, \ldots, S_n]$ where $S_i \in \mathcal{S}_i$ for all $i \in [n]$
- Generate γ_{k+1} from γ_k using $S[|\mathcal{I}_k|] = S_{|\mathcal{I}_k|}$

Strategy Formalization

- $S := [S_1, S_2, \ldots, S_n]$ where $S_i \in \mathcal{S}_i$ for all $i \in [n]$
- Generate γ_{k+1} from γ_k using $S[|\mathcal{I}_k|] = S_{|\mathcal{I}_k|}$

Example

$\gamma_2 = 5\mathbf{2}1\mathbf{3}4$, $\mathcal{I}_2 = \{1, 3, 5\} \implies$ use $S[3] \in \mathcal{S}_3$

Strategy Formalization

- $S := [S_1, S_2, \ldots, S_n]$ where $S_i \in \mathcal{S}_i$ for all $i \in [n]$
- Generate γ_{k+1} from γ_k using $S[|\mathcal{I}_k|] = S_{|\mathcal{I}_k|}$

Example

$\gamma_2 = 5\mathbf{2}1\mathbf{3}4$, $\mathcal{I}_2 = \{1, 3, 5\} \implies$ use $S[3] \in \mathcal{S}_3$

- If $S[3] = 231$, then $514 \mapsto 451$
  $\hookrightarrow \gamma_3 = 42531$

Strategy Formalization

- $S := [S_1, S_2, \ldots, S_n]$ where $S_i \in \mathcal{S}_i$ for all $i \in [n]$
- Generate γ_{k+1} from γ_k using $S[|\mathcal{I}_k|] = S_{|\mathcal{I}_k|}$

Example

$\gamma_2 = 5\mathbf{2}1\mathbf{3}4$, $\mathcal{I}_2 = \{1, 3, 5\} \implies$ use $S[3] \in \mathcal{S}_3$

- If $S[3] = 231$, then $514 \mapsto 451$
  $\hookrightarrow \gamma_3 = 42531$

Alternative:

- If $S[3] = 132$, then $514 \mapsto 541$
  $\hookrightarrow \gamma_3 = 52431$

Cyclic Shift & Other Strategies

Cyclic Shifts

$CS := [(1), (21), (231), \ldots, (2\ 3\ \ \ldots\ \ n\ 1)]$
$CS' := [(1), (21), (312), \ldots, (n\ 1\ 2\ \ \ldots\ \ n-1)]$

Cyclic Shift & Other Strategies

Cyclic Shifts

$CS := [(1), (21), (231), \ldots, (2\ 3\ \ldots\ n\ 1)]$
$CS' := [(1), (21), (312), \ldots, (n\ 1\ 2\ \ldots\ n-1)]$

Cyclic Strategies:
$S := [C_1, C_2, C_3, \ldots, C_n]$ where C_i cyclic perm of $[i]$

Cyclic Shift & Other Strategies

Cyclic Shifts

$CS := [(1), (21), (231), \ldots, (2\ 3\ \ldots\ n\ 1)]$
$CS' := [(1), (21), (312), \ldots, (n\ 1\ 2\ \ldots\ n-1)]$

Cyclic Strategies:
$S := [C_1, C_2, C_3, \ldots, C_n]$ where C_i cyclic perm of $[i]$
$\implies S := [(1), (21), C_3, \ldots, C_n]$

Cyclic Shift & Other Strategies

Cyclic Shifts

$CS := [(1), (21), (231), \ldots, (2\ 3\ \ldots\ n\ 1)]$
$CS' := [(1), (21), (312), \ldots, (n\ 1\ 2\ \ldots\ n-1)]$

Cyclic Strategies:

$S := [C_1, C_2, C_3, \ldots, C_n]$ where C_i cyclic perm of $[i]$

$\implies S := [(1), (21), C_3, \ldots, C_n]$

$\implies C_3 = (231)$ or (312)

Cyclic Shift & Other Strategies

Cyclic Shifts

$CS := [(1), (21), (231), \ldots, (2\ 3\ \ldots\ n\ 1)]$
$CS' := [(1), (21), (312), \ldots, (n\ 1\ 2\ \ldots\ n-1)]$

Cyclic Strategies:
$S := [C_1, C_2, C_3, \ldots, C_n]$ where C_i cyclic perm of $[i]$
$\implies S := [(1), (21), C_3, \ldots, C_n]$
$\implies C_3 = (231)$ or (312)

Motivation:

- No fixed points
- Smaller scope
- Easier to generate/analyze, recursive patterns

Cyclic Shift & Other Strategies

Cyclic Shifts

$CS := [(1), (21), (231), \ldots, (2\ 3\ \ldots\ n\ 1)]$
$CS' := [(1), (21), (312), \ldots, (n\ 1\ 2\ \ldots\ n-1)]$

Cyclic Strategies:

$S := [C_1, C_2, C_3, \ldots, C_n]$ where C_i cyclic perm of $[i]$
$\implies S := [(1), (21), C_3, \ldots, C_n]$
$\implies C_3 = (231)$ or (312)

Motivation:

- No fixed points
- Smaller scope
- Easier to generate/analyze, recursive patterns

Later: $S := [D_1, D_2, D_3, \ldots, D_n]$ where $D_i \in Der(i)$

Generating Functions

$f_S(x) = \sum_{m \geq 1} a_m x^m$ where $a_m = \#\pi \in \mathcal{S}_n$ guessed in m guesses by strategy S

★ Notation: $[x^m]f_S(x) = a_m$

Example: $S = [(1), (21), (231)]$ has $f_S(x) = x + 4x^2 + x^3$

Generating Functions

$f_S(x) = \sum_{m \geq 1} a_m x^m$ where $a_m = \#\pi \in \mathcal{S}_n$ guessed in m guesses by strategy S

★ Notation: $[x^m] f_S(x) = a_m$

Example: $S = [(1), (21), (231)]$ has $f_S(x) = x + 4x^2 + x^3$

Remarks:

- Maximum degree not fixed (bounded?)
- Coeffs of $f_{CS}(x)$ are Eulerian numbers! (Kutin-Smithline)
- For any S, $[x^1] f_S(x) = 1$ (trivial perm.)
- For any S, $[x^2] f_S(x)$ is fixed (let's prove it!)
- $[x^3] f_{CS}(x) > [x^3] f_S(x)$ for any other S
  $\hookrightarrow$ By induction! See Hiveley 2025, Section 4

One (Mandatory) Proof

Theorem

For all strategies S of length n, $[x^2]f_S(x) = A(n,1)$

Proof.

□

One (Mandatory) Proof

Theorem

For all strategies S of length n, $[x^2]f_S(x) = A(n,1)$

Proof.

WLOG, $\gamma_1 = (1\ 2\ \ldots\ n)$. Have some set $\mathcal{J}_1$ of correct entries, then γ_2 is deterministic, and game ends!

□

One (Mandatory) Proof

Theorem

For all strategies S of length n, $[x^2]f_S(x) = A(n,1)$

Proof.

WLOG, $\gamma_1 = (1\ 2\ \ldots\ n)$. Have some set $\mathcal{J}_1$ of correct entries, then γ_2 is deterministic, and game ends!

- Count γ_2's by counting legal subsets $\mathcal{J}_1 \subset [n]$

□

One (Mandatory) Proof

Theorem

For all strategies S of length n, $[x^2]f_S(x) = A(n,1)$

Proof.

WLOG, $\gamma_1 = (1\ 2\ \ldots\ n)$. Have some set $\mathcal{J}_1$ of correct entries, then γ_2 is deterministic, and game ends!

- Count γ_2's by counting legal subsets $\mathcal{J}_1 \subset [n]$
- $|\mathcal{J}_1| \neq n, n-1$ (else end in one guess!) $\implies 0 \leq |\mathcal{J}_1| \leq n-2$

□

One (Mandatory) Proof

Theorem

For all strategies S of length n, $[x^2]f_S(x) = A(n, 1)$

Proof.

WLOG, $\gamma_1 = (1\ 2\ \ldots\ n)$. Have some set $\mathcal{J}_1$ of correct entries, then γ_2 is deterministic, and game ends!

- Count γ_2's by counting legal subsets $\mathcal{J}_1 \subset [n]$
- $|\mathcal{J}_1| \neq n, n-1$ (else end in one guess!) $\implies 0 \leq |\mathcal{J}_1| \leq n-2$
- If $|\mathcal{J}_1| = k$, then there are $\binom{n}{k}$ ways to select correct entries

□

One (Mandatory) Proof

Theorem

For all strategies S of length n, $[x^2]f_S(x) = A(n,1)$

Proof.

WLOG, $\gamma_1 = (1\ 2\ \dots\ n)$. Have some set $\mathcal{J}_1$ of correct entries, then γ_2 is deterministic, and game ends!

- Count γ_2's by counting legal subsets $\mathcal{J}_1 \subset [n]$
- $|\mathcal{J}_1| \neq n, n-1$ (else end in one guess!) $\implies 0 \leq |\mathcal{J}_1| \leq n-2$
- If $|\mathcal{J}_1| = k$, then there are $\binom{n}{k}$ ways to select correct entries
- $[x^2]f_S(x) = \#\mathcal{J}_1 = \sum_{k=0}^{n-2} \binom{n}{k} = A(n,1)$ (A000295)

□

Cubic Coefficients

More complicated. When is our first entry correct?

Cubic Coefficients

More complicated. When is our first entry correct?

- $\mathcal{J}_1 \neq \emptyset \to$ similar to before!
- $\mathcal{J}_3 \neq \emptyset \implies$ one such perm.

Cubic Coefficients

More complicated. When is our first entry correct?

- $\mathcal{J}_1 \neq \emptyset \to$ similar to before!
- $\mathcal{J}_3 \neq \emptyset \implies$ one such perm.
- $\mathcal{J}_2 \neq \emptyset$. Harder to count. Be careful not to backtrack incorrectly:

Cubic Coefficients

More complicated. When is our first entry correct?

- $\mathcal{J}_1 \neq \emptyset \rightarrow$ similar to before!
- $\mathcal{J}_3 \neq \emptyset \implies$ one such perm.
- $\mathcal{J}_2 \neq \emptyset$. Harder to count. Be careful not to backtrack incorrectly:

Example: For CS with $n = 6$:

123456	$\mathcal{J}_1 = \emptyset$
612345	$\mathcal{J}_2 = \{1, 2, 5, 6\}$
613245	BAD! $\pi[3] \neq 3$

Cubic Coefficients

More complicated. When is our first entry correct?

- $\mathcal{J}_1 \neq \emptyset \to$ similar to before!
- $\mathcal{J}_3 \neq \emptyset \implies$ one such perm.
- $\mathcal{J}_2 \neq \emptyset$. Harder to count. Be careful not to backtrack incorrectly:

Example: For CS with $n = 6$:

123456	$\mathcal{J}_1 = \emptyset$
612345	$\mathcal{J}_2 = \{1, 2, 5, 6\}$
613245	BAD! $\pi[3] \neq 3$

Count $\mathcal{J}_2 \subset [n]$ s.t.:

- $|\mathcal{J}_2| \leq n - 2$ (cannot be n or $n - 1$ to end in 3 guesses)
- $\mathcal{I}_2 \neq \{i, i + 1\}$ for any $i \pmod{n}$ $\quad$ $*$ *for CS, specifically*

Suboptimality in Duplication

Duplicated information $\implies$ BAD!

Suboptimality in Duplication

Duplicated information $\implies$ BAD!
Example: $S[4] = 2143$, $\pi = 3421$

$$\begin{array}{ll} \gamma_1 = 1234 & \mathcal{J}_1 = \emptyset \\ \gamma_2 = 2143 & \mathcal{J}_2 = \emptyset \\ \gamma_3 = 1234 & \mathcal{J}_3 = \emptyset \\ \gamma_4 = 2143 & \mathcal{J}_4 = \emptyset \\ \vdots & \vdots \end{array}$$

Suboptimality in Duplication

Duplicated information $\implies$ BAD!
Example: $S[4] = 2143$, $\pi = 3421$

$$\begin{array}{ll} \gamma_1 = 1234 & \mathcal{J}_1 = \emptyset \\ \gamma_2 = 2143 & \mathcal{J}_2 = \emptyset \\ \gamma_3 = 1234 & \mathcal{J}_3 = \emptyset \\ \gamma_4 = 2143 & \mathcal{J}_4 = \emptyset \\ \vdots & \vdots \end{array}$$

Observations:

- Clearly can happen for non-cyclic derangements.
- True whenever cycle lengths pair off (see Hiveley 2026, Theorem 6)

Duplication (cont.)

Ex: $S = [(1),(21),(231),(2143),(34521)]$ w/ $\pi = 41523$

$$\begin{aligned} \gamma_1 &= 12345 & \mathcal{J}_1 &= \emptyset \\ \gamma_2 &= 54123 & \mathcal{J}_2 &= \{4,5\} \\ \gamma_3 &= 15423 & \mathcal{J}_3 &= \{4,5\} \\ \gamma_4 &= 41523 & \mathcal{J}_4 &= \{1,2,3,4,5\} \end{aligned}$$

Duplication (cont.)

Ex: $S = [(1),(21),(231),(2143),(34521)]$ w/ $\pi = 41523$

$$\begin{array}{ll} \gamma_1 = 12345 & \mathcal{J}_1 = \emptyset \\ \gamma_2 = 54123 & \mathcal{J}_2 = \{4,5\} \\ \gamma_3 = 15423 & \mathcal{J}_3 = \{4,5\} \\ \gamma_4 = 41523 & \mathcal{J}_4 = \{1,2,3,4,5\} \end{array}$$

$*$ Entry 1 is guessed incorrectly in position one in both γ_1 *and* γ_3

Duplication (cont.)

Ex: $S = [(1), (21), (231), (2143), (34521)]$ w/ $\pi = 41523$

$$\begin{array}{ll} \gamma_1 = 12345 & \mathcal{J}_1 = \emptyset \\ \gamma_2 = 54123 & \mathcal{J}_2 = \{4, 5\} \\ \gamma_3 = 15423 & \mathcal{J}_3 = \{4, 5\} \\ \gamma_4 = 41523 & \mathcal{J}_4 = \{1, 2, 3, 4, 5\} \end{array}$$

$*$ Entry 1 is guessed incorrectly in position one in both γ_1 *and* γ_3

Theorem

CS never duplicates

Intuitively, sure! Proof in Kutin & Smithline

Duplication (cont.)

Ex: $S = [(1), (21), (231), (2143), (34521)]$ w/ $\pi = 41523$

$$\begin{array}{ll} \gamma_1 = 12345 & \mathcal{J}_1 = \emptyset \\ \gamma_2 = 54123 & \mathcal{J}_2 = \{4,5\} \\ \gamma_3 = 15423 & \mathcal{J}_3 = \{4,5\} \\ \gamma_4 = 41523 & \mathcal{J}_4 = \{1,2,3,4,5\} \end{array}$$

$*$ Entry 1 is guessed incorrectly in position one in both γ_1 *and* γ_3

Theorem

CS never duplicates

Intuitively, sure! Proof in Kutin & Smithline

Theorem

Every other strategy does

Proof by nasty constructive algorithms (see Hiveley 2026)

What's Next?

- Coefficients of x^k for $k \geq 4$
- Analysis of proposed "worst" strategy:
  $CSL = [(1), (21), (231), \ldots, \underbrace{(n\ 1\ 2\ \ \ldots\ \ n-1)}_{\text{left shifting}}]$
- Prove Kutin & Smithline's conjecture!
- Extensions: Mastermind with Wordle-like feedback (R. Li & S. Zhu), multicolor permutation wordle (Kutin & Smithline)

Thanks & Questions